Solutions

to Selected Exercises

1.4. From Play to Power:

1. Late-night cash. Simplify the problem by considering a specific case. Suppose Dave and Paul both have \$100. If Dave gives Paul \$5, then Dave and Paul have \$95 and \$105 respectively. Problem solved. Note that the amount Dave must give Paul is unrelated to the initial amount of money in each of their pockets. Though the \$100 was extraneous information, it makes the problem easier to solve, and easier to explain!

2. Politicians on parade. If there were more than one honest politician, fact (b) would be false. The only possibility left is that there is exactly one honest politician; the remaining ninety-nine are crooked.

3. The profit. As in “Late-night cash,” we can clarify the problem by making it specific. Suppose that the dealer starts with \$20 (the initial amount doesn’t affect the final profit).

After…	First purchase	First sell	Second purchase	Second sell
	\$13	\$21	\$12	\$22

At the end of the transactions, she has made a profit of \$2.

4. The truth about… Let’s first consider the question without any limits on the number of pets. What’s the maximum possible number of cats? Because there are 56 biscuits and each cat takes 5 of them, there are at most 11 cats. Now explore the possibilities. If there were exactly 11 cats, then there would be one biscuit left over. If there were only 10 cats, then there would be six biscuits left over, exactly enough to feed one dog! That’s one solution, in theory, except that we are told there are 10 pets total. To find the only valid solution, continue exploring alternatives.

Number of Cats (5)	11	10	9	8	7	6	5	4	3	2	1	0
Number of Dogs (6)	0	1	1	2	3	4	5	6	6	7	8	9
Leftover Biscuits	1	0	5	4	3	2	1	0	5	4	3	2

The one solution is 4 cats and 6 dogs.

5. It’s in the box. There are two possible situations: (a) both signs are true, or (b) both are false. Let’s assume first that both signs are true. B’s sign says that box A contains a snake, and A’s sign says that at least one of the two boxes contains money. So in this situation, A contains a snake and B holds money. Now assume that both signs are false. If B’s sign is false, then box A contains money. If A’s sign is false, then both box A and box B must contain snakes. Because box A cannot contain both money and a snake, this whole situation is impossible – both signs can’t be false. Because the second alternative is impossible, both signs are true, and box B holds a million dollars.

6. Lights out. At first glance this problem seems impossible. When you walk into the other room, you will see one of two possibilities. Either the light is on, or the light is off. Two possible outcomes can’t resolve three possible situations! In order to solve the problem, we must find more information. The key to this problem is that real lights (most of them, anyway) become hot after being on for an extended period of time. Furthermore, once hot, they take a long time to cool down. So you can obtain extra information by turning on one of the switches, waiting ten minutes, and then turning off the switch.

Turn on switch #1. Wait ten minutes. Turn off switch #1. Now turn on switch #2 and quickly go into the other room and put your hand on the light. There are now three possible outcomes.

Outcome	**The lucky switch**
Light is on	#2
Light is off and cold	#3
Light is off and warm	#1

7. Out of sight but not out of mind. There is no mystery dollar. Slip paid $9, Spike paid $9, and Milly paid $7 for a total of $25, which was precisely the cost of the room. Chip's mistake lies in his addition of Milly's stolen money to the $27. The corrected logic: Each band member puts in $10, for a total of $30. Where does this $30 go? $1 worth of change goes to each of the band members, leaving $27 to be accounted for. Milly pocketed 2 of these dollars and Chip took the remaining $25 for the cost of the room. All is accounted for.

8. Comedy Central.

	River	
CCCOOO <>		
COOO		<>CC
CCOOO<>		C
OOO		<>CCC
COOO<>		CC
CO		<>CCOO
CCOO<>		CO
CC		<>COOO
CCC<>		OOO
C		<>CCOOO
CC<>		COOO
		<>CCCOOO

C = Comedian, O = Officer, <> = Boat

There are several solutions to this problem. Notice the symmetry in the above solution; the first five trips are identical to the last five trips (in reverse order).

9. Whom do you trust? Pocket states that Shlock and Wind are lying. Because there can be at most two liars, this would imply that the remaining three (Pocket, Greede, and Slie) are all telling the truth. Because Greede and Slie contradict each other, this is not possible. So we know for sure that Pocket is lying. There is room for at most one more liar, and once again, because Greede and Slie contradict each other, one of them is the other liar. It doesn't matter which one is the liar because by now we know that both Shlock and Wind are telling the truth. Shlock says that either Wind or Pocket leaked the story, and Wind denies leaking it himself. So it was Pocket who leaked the story.

10. A commuter fly. The trains are traveling towards each other at a rate of 70 miles per hour. Because they were initially 210 miles apart, the trains will collide in 3 hours. Because the fly travels at 100 miles per hour, the fly travels a total of 300 miles.

11. Taxing taxi. [or… A fair fare…] " How much should each person pay?" The term 'should' isn't as precise as it sounds. We'll present two methods for divvying up the fare and let you be the judge. (1) Everyone should pay the same mileage rate. That is, because Mary travels twice as far as Dave does, she should expect to pay twice as much. There are 60 total passenger miles (10+20+30), and a total cost is \$45. So each person pays \$0.75 a mile. Bob pays \$7.50, Mary pays \$15, and Ivan pays \$22.50. Everyone pays exactly half of what they would have paid had they traveled alone. (2) The fare for each mile should be split evenly among the passengers. All three passengers split the fare for the first ten miles. Mary and Ivan split the fare for the second ten miles, and Ivan pays for the last ten miles himself. Bob pays \$5, Mary pays \$12.50, and Ivan pays \$27.50. Ivan pays more per mile on average than Bob, but then again, Bob had to share his whole trip cramped with three other people, while Ivan spent the last ten miles of his trip with plenty of room to stretch.

12. Getting a pole on a bus. Sarah gave Adam a box that was 4 feet long and 3 feet wide. The diagonal of the box was five feet, just long enough to store Adam's fishing pole. (Recall from the Pythagorean theorem that the square of the length of the diagonal is the sum of the squares of the two sides; $5^2 = 3^2 + 4^2$.)

13. Tea time. Just as in "A commuter fly", there is a long, arithmetic-laden solution and a short insightful solution. We started with 3 oz of cream in the creamer, and 3 oz of tea in the teacup. In the end, we still have 3 oz of liquid in both containers, but we don't know the relative amounts of tea and cream. The arithmetic-laden solution tries to find out exactly how much of each liquid is in each cup. We can avoid this by arguing only that there is as much cream in the teacup as there is tea in the creamer.

Suppose there are C oz of cream in the teacup. Because the teacup holds 3 oz, there are (3 – C) oz of tea also in the teacup. Because we started with a total of 3 oz of tea, the remaining C oz of tea must be in the creamer. So, the teacup is just as diluted as the creamer.

14. A shaky story. Let's refer to Sam's guests by the number of hands they shook. Specifically, G0 shook no hands, G1 shook one hand, up to G8, who shook eight other hands. Because there are 10 guests in all, G8 must have shaken hands with everyone but himself and G0. Because G8 did not shake hands with his spouse, the only person who could possibly be his spouse is G0. So, G0 and G8 are married. With this new information, we can simplify the problem.

The four remaining couples consist of the following eight people: Sam, G1, G2, G3, G4, G5, G6, and G7. Note that G8 shook hands with everyone in this new group. Because G7 shook hands with G8, he must have shaken hands with exactly six members of the new group. Similarly, G1 shook hands with no one else. We are in the same situation as before. G7 didn't shake hands with G1, and because he shook hands with everyone else, G1 is the only possible candidate for G7's spouse. So, G1 and G7 are married.

Continue simplifying the problem. Sam, G2, G3, G4, G5, and G6 all shook hands with both G7 and G8. A similar line of reasoning can show that G2 and G6 are married. A final simplification allows us to conclude that G3 and G5 are married. The remaining couple consists of Sam and G4. So Sam's wife shook exactly four hands. Is it possible to determine how many hands Sam shook?

15. Murray's brother. Begin by placing four stones on each side of the balance (see the first table). There are three possible outcomes indicated by the three columns in the table. Either the left side is heavier, the right side is heavier, or the scale balances. The first two outcomes leave us in the same situation–we know the diamond is among eight of the stones, and if we knew which stone it was, we would also know whether the diamond was heavier or lighter. (The letters H and L below indicate this.) If the scale balances, then we know the diamond is one of the four remaining stones.

Table 1: First Weighing: ???? vs. ???? Extra Stones: ????

Possible Outcomes	HHHH vs. LLLL extra stones: SSSS	LLLL vs. HHHH extra stones: SSSS	SSSS vs. SSSS extra stones: ????

?, unknown stone; H, potentially heavy stone; L, potentially light stone; D, dormant diamond; S, stone not containing diamond.

The second and third tables outline the argument needed to address these two situations. The last row in each table represents the third weighing needed for each situation. For each outcome of each weighing, make sure that you can identify the diamond and its relative weight.

Table 2: Second and Third Weighings When First Weighing Balances: SSSS vs. SSSS with ???? extra stones

Second weighing		SSS vs. ??? extra stone: ?	
Possible Outcomes	SSS vs. LLL extra: S (The right side of the scale is lighter, thus D lies in one of the L stones, which is lighter than all the rest.)	SSS vs. HHH extra: S (The right side of the scale is lighter, thus D lies in one of the H stones, which is heavier than all the rest.)	SSS vs. SSS extra: D (D is in the extra stone.)
Third weighing	L vs. L extra: L (If the scale balances, D is in the extra stone. Otherwise, it is in the stone on the lighter side of the scale.)	H vs. H extra: H (If the scale balances, D is in the extra stone. Otherwise, it is in the stone on the heavier side of the scale.)	S vs. D (This weighing will determine if the stone containing D is heavier or lighter.)

?, unknown stone; H, potentially heavy stone; L, potentially light stone; D, dormant diamond; S, stone not containing diamond.

Table 3: Second and Third Weighings When First Weighing Does Not Balance: HHHH vs. LLLL or LLLL vs. HHHH with SSSS extra stones

Second weighing		HHL vs. HHL extra LL	
Possible Outcomes	HHS vs. SSL extra: SS (The left side is heavier, so either D must be in one of the two Hs on the left side, which would be heavier by the first weighing, or D is in the L stone on the right.)	SSL vs. HHS extra: SS (The right side is heavier, so either D must be in one of the two Hs on the right side, which would be heavier by the first weighing, or D is in the L stone on the left.)	SSS vs. SSS extra: LL (The scales balance, so D lies in one of the extras, which we now know is lighter from the first weighing.)
Third weighing	H vs. H extra: L (If the scales balance, D is in the extra L; otherwise, D is in the heavier H.)	H vs. H extra: L (If the scales balance, D is in the extra L; otherwise, D is in the heavier H.)	L vs. L (D lies in the lighter stone.)

?, unknown stone; H, potentially heavy stone; L, potentially light stone; D, dormant diamond; S, stone not containing diamond.

16. Cutting (chess) boards. For the warm up question, we can easily cover the 8x8 checkerboard with 32 dominoes: use four horizontal dominoes for each row. To cover the board with the top two corners removed, just remove one domino from the top row and slide the remaining dominoes over appropriately. For the last challenge, look carefully at the truncated checkerboard. We see that both deleted squares are black. This leaves 62 squares to cover with dominoes: 32 white squares and 30 black squares. Yet each domino will cover exactly one black square and one white square, so any domino arrangement will cover the same number of whites squares as black squares. This there is no way to cover the truncated board with dominoes.

17. Siegfried & You. The deck contains 26 black cards and 26 red cards to begin with. If the first pile of 26 cards contains n black cards, then it contains $26 - n$ red cards. Thus, the remaining n red cards must be in the other pile.

18. Penny for your thoughts. Suppose there are exactly k coins showing heads. Divide the coins arbitrarily into two collections so that one collection contains k coins. The collection with k coins contains some unknown number of heads, say i, with the remaining $k - i$ coins showing tails. Note that the second collection must have $k - i$ heads. Now turn over all the coins in the first collection. The i heads become tails and the $k - i$ tails become heads, which is the exact number of heads in the second collection.

19. When will the world end? We start at the beginning, assuming that the disks are all on the left peg. For one disk, clearly one move suffices. For two disks, we move the top disk to the right peg, then the bottom disk to the middle peg, then the top disk to the middle peg. So, two disks require three moves. Notice that we just as easily could have used three similar moves to transfer the stack of two disks to the right peg instead of to the middle peg. In fact, we require three moves to transfer two disks from one peg to any other peg.

To move three disks from the left peg to the middle peg, first use three moves to move the top two disks to the right peg, then move the bottom disk to the middle peg, then use three more moves to move the top two disks to the middle peg. Thus, three disks require a total of seven moves. We can use this approach for four disks as well. Move the top three disks to the right peg in seven moves, then move the bottom disk to the middle peg, then move the top three disks to the middle peg in seven more moves, for a total of 15 moves.

If we let $M(k)$ equal to number of moves needed for k disks, then we have just shown that $M(4) = 2\times M(3) + 1$. This relation holds in general. If we have k disks on the left peg, we can't move the bottom disk to the middle peg until all the disks on top of it have been moved to the right peg. We require $M(k - 1)$ moves to transfer these disks to the right peg. Then one move to transfer the bottom disk to the middle peg, followed by $M(k - 1)$ moves to transfer the remaining disks to the middle peg. Thus, we have $M(k) = 2\times M(k - 1) + 1$, when we start with $M(1) = 1$. This gives us the following values:

$M(1) = 1$
$M(2) = 2\times M(1) + 1 = 2\times 1 + 1 = 3$
$M(3) = 2\times M(2) + 1 = 2\times 3 + 1 = 7$
$M(4) = 2\times M(3) + 1 = 2\times 7 + 1 = 15$
$M(5) = 2\times M(4) + 1 = 2\times 15 + 1 = 31$
$M(6) = 2\times M(5) + 1 = 2\times 31 + 1 = 63$.

We notice that each value is one less than a power of 2, so we can conjecture that

$M(k) = 2^k - 1$. This formula is valid for $k = 1, 2, 3, \ldots$, and can be proven so using mathematical induction.

Now what about the monks and the end of the world? Each move requires one second, so moving 64 disks requires $2^{64} - 1$ seconds. There are about $60\times 60\times 24\times 365 = 31{,}536{,}000$ seconds in a year, so the monks' task will be completed in about $(2^{64} - 1)/31{,}536{,}000$ years. This is about 5.85×10^{12} years, so it appears we still have some time to enjoy our planet.

20. The fork in the road. Here's a question to which both Liars and Truth-Tellers will give the same answer: "Which road will a member of the *other* tribe tell me leads to the villa?" Here's how the table would be completed under the assumption that the left road leads to the villa. Note that both answers are the same, and you know to take the other road.

	"Which road leads to the villa?"	"Which road will a member of the *other* tribe tell me leads to the villa?"
Answer from Liar	RIGHT	RIGHT
Answer from Truth-Teller	LEFT	RIGHT