CHAPTER 2: DETERMINANTS

2.1 Determinants by Cofactor Expansion

2.

$M_{11} = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 3 & 6 \\ 0 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 3 & 6 \\ 1 & 4 \end{vmatrix} = 6$ $\qquad C_{11} = (-1)^{1+1}M_{11} = M_{11} = 6$

$M_{12} = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 3 & 6 \\ 0 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 3 & 6 \\ 0 & 4 \end{vmatrix} = 12$ $\qquad C_{12} = (-1)^{1+2}M_{12} = -M_{12} = -12$

$M_{13} = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 3 & 6 \\ 0 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 3 & 3 \\ 0 & 1 \end{vmatrix} = 3$ $\qquad C_{13} = (-1)^{1+3}M_{13} = M_{13} = 3$

$M_{21} = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 3 & 6 \\ 0 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = 2$ $\qquad C_{21} = (-1)^{2+1}M_{21} = -M_{21} = -2$

$M_{22} = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 3 & 6 \\ 0 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} = 4$ $\qquad C_{22} = (-1)^{2+2}M_{22} = M_{22} = 4$

$M_{23} = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 3 & 6 \\ 0 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$ $\qquad C_{23} = (-1)^{2+3}M_{23} = -M_{23} = -1$

$M_{31} = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 3 & 6 \\ 0 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 0$ $\qquad C_{31} = (-1)^{3+1}M_{31} = M_{31} = 0$

$M_{32} = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 3 & 6 \\ 0 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 0$ $\qquad C_{32} = (-1)^{3+2}M_{32} = -M_{32} = 0$

$M_{33} = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 3 & 6 \\ 0 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix} = 0$ $\qquad C_{33} = (-1)^{3+3}M_{33} = M_{33} = 0$

4. (a) $M_{32} = \begin{vmatrix} 2 & -1 & 1 \\ -3 & 0 & 3 \\ 3 & 1 & 4 \end{vmatrix} = 2\begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix} - (-1)\begin{vmatrix} -3 & 3 \\ 3 & 4 \end{vmatrix} + 1\begin{vmatrix} -3 & 0 \\ 3 & 1 \end{vmatrix}$ ← cofactor expansion along the first row

$= 2(-3) + 1(-21) + 1(-3) = -30$

$C_{32} = (-1)^{3+2}M_{32} = -M_{32} = 30$

(b) $M_{44} = \begin{vmatrix} 2 & 3 & -1 \\ -3 & 2 & 0 \\ 3 & -2 & 1 \end{vmatrix} = 2\begin{vmatrix} 2 & 0 \\ -2 & 1 \end{vmatrix} - 3\begin{vmatrix} -3 & 0 \\ 3 & 1 \end{vmatrix} + (-1)\begin{vmatrix} -3 & 2 \\ 3 & -2 \end{vmatrix}$ ← cofactor expansion along the first row

$= 2(2) - 3(-3) - 1(0) = 13$

$C_{44} = (-1)^{4+4}M_{44} = M_{44} = 13$

(c) $M_{41} = \begin{vmatrix} 3 & -1 & 1 \\ 2 & 0 & 3 \\ -2 & 1 & 0 \end{vmatrix} = 3\begin{vmatrix} 0 & 3 \\ 1 & 0 \end{vmatrix} - (-1)\begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix} + 1\begin{vmatrix} 2 & 0 \\ -2 & 1 \end{vmatrix}$ ← cofactor expansion along the first row

$= 3(-3) + 1(6) + 1(2) = -1$

$C_{41} = (-1)^{4+1}M_{41} = -M_{41} = 1$

(d) $M_{24} = \begin{vmatrix} 2 & 3 & -1 \\ 3 & -2 & 1 \\ 3 & -2 & 1 \end{vmatrix} = 2\begin{vmatrix} -2 & 1 \\ -2 & 1 \end{vmatrix} - 3\begin{vmatrix} 3 & 1 \\ 3 & 1 \end{vmatrix} + (-1)\begin{vmatrix} 3 & -2 \\ 3 & -2 \end{vmatrix}$ ← cofactor expansion along the first row

$= 2(0) - 3(0) - 1(0) = 0$

$C_{24} = (-1)^{2+4}M_{24} = M_{24} = 0$

6. $\begin{vmatrix} 4 & 1 \\ 8 & 2 \end{vmatrix} = (4)(2) - (1)(8) = 0;$ The matrix is not invertible.

8. $\begin{vmatrix} \sqrt{2} & \sqrt{6} \\ 4 & \sqrt{3} \end{vmatrix} = (\sqrt{2})(\sqrt{3}) - (\sqrt{6})(4) = \sqrt{6} - 4\sqrt{6} = -3\sqrt{6} \neq 0.$ Inverse: $\frac{1}{-3\sqrt{6}}\begin{bmatrix} \sqrt{3} & -\sqrt{6} \\ -4 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{-1}{3\sqrt{2}} & \frac{1}{3} \\ \frac{4}{3\sqrt{6}} & \frac{-1}{3\sqrt{3}} \end{bmatrix}$

10. $\begin{vmatrix} -2 & 7 & 6 \\ 5 & 1 & -2 \\ 3 & 8 & 4 \end{vmatrix} = \begin{vmatrix} -2 & 7 & 6 \\ 5 & 1 & -2 \\ 3 & 8 & 4 \end{vmatrix}\begin{matrix} -2 & 7 \\ 5 & 1 \\ 3 & 8 \end{matrix} = [-8 - 42 + 240] - [18 + 32 + 140] = 0$

12. $\begin{vmatrix} -1 & 1 & 2 \\ 3 & 0 & -5 \\ 1 & 7 & 2 \end{vmatrix} = \begin{vmatrix} -1 & 1 & 2 \\ 3 & 0 & -5 \\ 1 & 7 & 2 \end{vmatrix}\begin{matrix} -1 & 1 \\ 3 & 0 \\ 1 & 7 \end{matrix} = [0 - 5 + 42] - [0 + 35 + 6] = -4$

14. $\begin{vmatrix} c & -4 & 3 \\ 2 & 1 & c^2 \\ 4 & c-1 & 2 \end{vmatrix} = \begin{vmatrix} c & -4 & 3 \\ 2 & 1 & c^2 \\ 4 & c-1 & 2 \end{vmatrix}\begin{matrix} c & -4 \\ 2 & 1 \\ 4 & c-1 \end{matrix} = [2c - 16c^2 + 6(c-1)] - [12 + (c-1)c^3 - 16]$

$= 2c - 16c^2 + 6c - 6 - 12 - c^4 + c^3 + 16 = -c^4 + c^3 - 16c^2 + 8c - 2$

16. Calculate the determinant by a cofactor expansion along the first row:

$$\det(A) = \begin{vmatrix} \lambda-4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda-1 \end{vmatrix} = (\lambda-4)\begin{vmatrix} \lambda & 2 \\ 3 & \lambda-1 \end{vmatrix} - 0 + 0$$

$$= (\lambda-4)[\lambda(\lambda-1)-6] = (\lambda-4)[\lambda^2-\lambda-6] = (\lambda-4)(\lambda-3)(\lambda+2)$$

The determinant is zero if $\lambda = -2$, $\lambda = 3$, or $\lambda = 4$.

18. Calculate the determinant by a cofactor expansion along the third row:

$$\det(A) = \begin{vmatrix} \lambda-4 & 4 & 0 \\ -1 & \lambda & 0 \\ 0 & 0 & \lambda-5 \end{vmatrix} = 0 - 0 + (\lambda-5)\begin{vmatrix} \lambda-4 & 4 \\ -1 & \lambda \end{vmatrix}$$

$$= (\lambda-5)[(\lambda-4)\lambda+4] = (\lambda-5)[\lambda^2-4\lambda+4] = (\lambda-5)(\lambda-2)^2$$

The determinant is zero if $\lambda = 2$ or $\lambda = 5$.

20. **(a)** $(-1)\begin{vmatrix} 0 & -5 \\ 7 & 2 \end{vmatrix} - 1\begin{vmatrix} 3 & -5 \\ 1 & 2 \end{vmatrix} + 2\begin{vmatrix} 3 & 0 \\ 1 & 7 \end{vmatrix} = (-1)(35) - 1(11) + 2(21) = -4$

(b) $(-1)\begin{vmatrix} 0 & -5 \\ 7 & 2 \end{vmatrix} - 3\begin{vmatrix} 1 & 2 \\ 7 & 2 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 0 & -5 \end{vmatrix} = (-1)(35) - 3(-12) + 1(-5) = -4$

(c) $-3\begin{vmatrix} 1 & 2 \\ 7 & 2 \end{vmatrix} + 0 - (-5)\begin{vmatrix} -1 & 1 \\ 1 & 7 \end{vmatrix} = -3(-12) + 0 + 5(-8) = -4$

(d) $-1\begin{vmatrix} 3 & -5 \\ 1 & 2 \end{vmatrix} + 0 - 7\begin{vmatrix} -1 & 2 \\ 3 & -5 \end{vmatrix} = -1(11) + 0 - 7(-1) = -4$

(e) $1\begin{vmatrix} 1 & 2 \\ 0 & -5 \end{vmatrix} - 7\begin{vmatrix} -1 & 2 \\ 3 & -5 \end{vmatrix} + 2\begin{vmatrix} -1 & 1 \\ 3 & 0 \end{vmatrix} = 1(-5) - 7(-1) + 2(-3) = -4$

(f) $2\begin{vmatrix} 3 & 0 \\ 1 & 7 \end{vmatrix} - (-5)\begin{vmatrix} -1 & 1 \\ 1 & 7 \end{vmatrix} + 2\begin{vmatrix} -1 & 1 \\ 3 & 0 \end{vmatrix} = 2(21) + 5(-8) + 2(-3) = -4$

22. Calculate the determinant by a cofactor expansion along the second row:

$$-1\begin{vmatrix} 3 & 1 \\ -3 & 5 \end{vmatrix} + 0 - (-4)\begin{vmatrix} 3 & 3 \\ 1 & -3 \end{vmatrix} = -1(18) + 0 + 4(-12) = -66$$

24. Calculate the determinant by a cofactor expansion along the second column:

$$-(k-1)\begin{vmatrix} 2 & 4 \\ 5 & k \end{vmatrix} + (k-3)\begin{vmatrix} k+1 & 7 \\ 5 & k \end{vmatrix} - (k+1)\begin{vmatrix} k+1 & 7 \\ 2 & 4 \end{vmatrix}$$

$$= -(k-1)(2k-20) + (k-3)\big((k+1)k-35\big) - (k+1)(4(k+1)-14)$$

$$= k^3 - 8k^2 - 10k + 95$$

26. Calculate the determinant by a cofactor expansion along the first row:

$$\det(A) = 4\begin{vmatrix} 3 & 3 & -1 & 0 \\ 2 & 4 & 2 & 3 \\ 4 & 6 & 2 & 3 \\ 2 & 4 & 2 & 3 \end{vmatrix} - 0 + 0 - 1\begin{vmatrix} 3 & 3 & 3 & 0 \\ 1 & 2 & 4 & 3 \\ 9 & 4 & 6 & 3 \\ 2 & 2 & 4 & 3 \end{vmatrix} + 0$$

Calculate each of the two determinants by a cofactor expansion along its first row:

$$\begin{vmatrix} 3 & 3 & -1 & 0 \\ 2 & 4 & 2 & 3 \\ 4 & 6 & 2 & 3 \\ 2 & 4 & 2 & 3 \end{vmatrix} = 3\begin{vmatrix} 4 & 2 & 3 \\ 6 & 2 & 3 \\ 4 & 2 & 3 \end{vmatrix} - 3\begin{vmatrix} 2 & 2 & 3 \\ 4 & 2 & 3 \\ 2 & 2 & 3 \end{vmatrix} + (-1)\begin{vmatrix} 2 & 4 & 3 \\ 4 & 6 & 3 \\ 2 & 4 & 3 \end{vmatrix} - 0 = 3(0) - 3(0) - 1(0) - 0 = 0$$

$$\begin{vmatrix} 3 & 3 & 3 & 0 \\ 1 & 2 & 4 & 3 \\ 9 & 4 & 6 & 3 \\ 2 & 2 & 4 & 3 \end{vmatrix} = 3\begin{vmatrix} 2 & 4 & 3 \\ 4 & 6 & 3 \\ 2 & 4 & 3 \end{vmatrix} - 3\begin{vmatrix} 1 & 4 & 3 \\ 9 & 6 & 3 \\ 2 & 4 & 3 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 & 3 \\ 9 & 4 & 3 \\ 2 & 2 & 3 \end{vmatrix} - 0 = 3(0) - 3(-6) + 3(-6) - 0 = 0$$

Therefore $\det(A) = 4(0) - 0 + 0 - 1(0) = 0$.

28. By Theorem 2.1.2, determinant of a diagonal matrix is the product of the entries on the main diagonal: $\det(A) = (2)(2)(2) = 8$.

30. By Theorem 2.1.2, determinant of an upper triangular matrix is the product of the entries on the main diagonal: $\det(A) = (1)(2)(3)(4) = 24$.

32. By Theorem 2.1.2, determinant of a lower triangular matrix is the product of the entries on the main diagonal: $\det(A) = (-3)(2)(-1)(3) = 18$

2.2 Evaluating Determinants by Row Reduction

2. $\det(A) = \det(A^T) = 10$

4. $\det(A) = \det(A^T) = 56$

6. By Theorem 2.1.2, determinant of a lower triangular matrix is the product of the entries on the main diagonal: $(1)(1)(1) = 1$.

8. By Theorem 2.1.2, determinant of a diagonal matrix is the product of the entries on the main diagonal: $(1)(-\frac{1}{3})(1)(1) = -\frac{1}{3}$.

10.

$$\begin{vmatrix} 3 & 6 & -9 \\ 0 & 0 & -2 \\ -2 & 1 & 5 \end{vmatrix} = 3\begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & -2 \\ -2 & 1 & 5 \end{vmatrix}$$

← A common factor of 3 from the first row was taken through the determinant sign.

$$= 3\begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & -2 \\ 0 & 5 & -1 \end{vmatrix}$$

← 2 times the first row was added to the third row.

$$= 3(-1)\begin{vmatrix} 1 & 2 & -3 \\ 0 & 5 & -1 \\ 0 & 0 & -2 \end{vmatrix}$$

← The second and third rows were interchanged.

$$= (3)(-1)(5)\begin{vmatrix} 1 & 2 & -3 \\ 0 & 1 & -\frac{1}{5} \\ 0 & 0 & -2 \end{vmatrix}$$

← A common factor of 5 from the second row was taken through the determinant sign.

$$= (3)(-1)(5)(-2)\begin{vmatrix} 1 & 2 & -3 \\ 0 & 1 & -\frac{1}{5} \\ 0 & 0 & 1 \end{vmatrix}$$

← A common factor of -2 from the last row was taken through the determinant sign.

$$= (3)(-1)(5)(-2)(1) = 30$$

12.

$$\begin{vmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 5 & -2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -3 & 0 \\ 0 & -2 & 1 \\ 5 & -2 & 2 \end{vmatrix}$$

← 2 times the first row was added to the second row.

$$= \begin{vmatrix} 1 & -3 & 0 \\ 0 & -2 & 1 \\ 0 & 13 & 2 \end{vmatrix}$$

← -5 times the first row was added to the third row.

$$= -2\begin{vmatrix} 1 & -3 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 13 & 2 \end{vmatrix}$$

← A common factor of -2 from the second row was taken through the determinant sign.

$$= -2\begin{vmatrix} 1 & -3 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{17}{2} \end{vmatrix}$$

← -13 times the second row was added to the third row.

$$= (-2)\left(\frac{17}{2}\right)\begin{vmatrix} 1 & -3 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{vmatrix}$$

← A common factor of $\frac{17}{2}$ from the last row was taken through the determinant sign.

$$= (-2)\left(\frac{17}{2}\right)(1) = -17$$

14. $\begin{vmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{vmatrix}$ ← -5 times the first row was added to the second row.

$= \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 2 & 8 & 6 & 1 \end{vmatrix}$ ← The first row was added to the third row.

$= \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 12 & 0 & -1 \end{vmatrix}$ ← -2 times the first row was added to the fourth row.

$= \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 108 & 23 \end{vmatrix}$ ← -12 times the second row was added to the fourth row.

$= -3\begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 108 & 23 \end{vmatrix}$ ← A common factor of -3 from the third row was taken through the determinant sign.

$= -3\begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & -13 \end{vmatrix}$ ← -108 times the third row was added to the fourth row.

$= (-3)(-13)\begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 1 \end{vmatrix}$ ← A common factor of -13 from the third row was taken through the determinant sign.

$= (-3)(-13)(1) = 39$

16. $\begin{vmatrix} 0 & 1 & 1 & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 & 0 \end{vmatrix} = -\begin{vmatrix} \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 1 & 1 & 1 \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 & 0 \end{vmatrix}$ ← The first and second rows were interchanged.

$= -\frac{1}{2}\begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 & 0 \end{vmatrix}$ ← A common factor of $\frac{1}{2}$ from the first row was taken through the determinant sign.

$= -\frac{1}{2}\begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -\frac{1}{3} & -1 & -\frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & 0 & 0 \end{vmatrix}$ ← $-\frac{2}{3}$ times the first row was added to the third row.

$$= -\frac{1}{2}\begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -\frac{1}{3} & -1 & -\frac{2}{3} \\ 0 & 1 & \frac{2}{3} & \frac{1}{3} \end{vmatrix}$$

← $\frac{1}{3}$ times the first row was added to the fourth row.

$$= -\frac{1}{2}\begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -\frac{2}{3} & -\frac{1}{3} \\ 0 & 1 & \frac{2}{3} & \frac{1}{3} \end{vmatrix}$$

← $\frac{1}{3}$ times the second row was added to the third row.

$$= -\frac{1}{2}\begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -\frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & -\frac{1}{3} & -\frac{2}{3} \end{vmatrix}$$

← -1 times the second row was added to the fourth row.

$$= \left(-\frac{1}{2}\right)\left(-\frac{2}{3}\right)\begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{3} & -\frac{2}{3} \end{vmatrix}$$

← A common factor of $-\frac{2}{3}$ from the third row was taken through the determinant sign.

$$= \left(-\frac{1}{2}\right)\left(-\frac{2}{3}\right)\begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & -\frac{1}{2} \end{vmatrix}$$

← $\frac{1}{3}$ times the third row was added to the fourth row.

$$= \left(-\frac{1}{2}\right)\left(-\frac{2}{3}\right)\left(-\frac{1}{2}\right)\begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

← A common factor of $-\frac{1}{2}$ from the last row was taken through the determinant sign.

$$= \left(-\frac{1}{2}\right)\left(-\frac{2}{3}\right)\left(-\frac{1}{2}\right)(1) = -\frac{1}{6}$$

18. Repeat Exercise 10:

$$\begin{vmatrix} 3 & 6 & -9 \\ 0 & 0 & -2 \\ -2 & 1 & 5 \end{vmatrix} = -(-2)\begin{vmatrix} 3 & 6 \\ -2 & 1 \end{vmatrix}$$

← Cofactor expansion along the second row.

$$= -(-2)\begin{vmatrix} 15 & 0 \\ -2 & 1 \end{vmatrix}$$

← -6 times the second row was added to the first row.

$$= -(-2)(15) = 30$$

Repeat Exercise 11:

$$\begin{vmatrix} 0 & 3 & 1 \\ 1 & 1 & 2 \\ 3 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 0 & 3 & 1 \\ 1 & 1 & 2 \\ 0 & -1 & -2 \end{vmatrix}$$

← -3 times the second row was added to the last row.

$$= -1 \begin{vmatrix} 3 & 1 \\ -1 & -2 \end{vmatrix}$$

← Cofactor expansion along the first column.

$$= (-1)(-5) = 5$$

Repeat Exercise 12:

$$\begin{vmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 5 & -2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 9 & -10 & 0 \end{vmatrix}$$

← -2 times the second row was added to the last row.

$$= -1 \begin{vmatrix} 1 & -3 \\ 9 & -10 \end{vmatrix}$$

← Cofactor expansion along the third column.

$$(-1)(17) = -17$$

Repeat Exercise 13:

$$\begin{vmatrix} 3 & -6 & 9 \\ -2 & 7 & -2 \\ 0 & 1 & 5 \end{vmatrix} = \begin{vmatrix} 3 & 0 & 39 \\ -2 & 0 & -37 \\ 0 & 1 & 5 \end{vmatrix}$$

← 6 times the last row was added to the first; -7 times the last row was added to the second row.

$$= -1 \begin{vmatrix} 3 & 39 \\ -2 & -37 \end{vmatrix}$$

← Cofactor expansion along the second column.

$$= (-1)(-33) = 33$$

20. The first and the third rows were interchanged, therefore $\begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix} = -\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -(-6) = 6.$

22. The third row is proportional to the first row, therefore by Theorem 2.2.5 $\begin{vmatrix} a & b & c \\ d & e & f \\ 2a & 2b & 2c \end{vmatrix} = 0.$

(This can also be shown by adding -2 times the first row to the third, then performing a cofactor expansion of the resulting determinant $\begin{vmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 0 \end{vmatrix}$ along the third row.)

24. $$\begin{vmatrix} a+d & b+e & c+f \\ -d & -e & -f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ -d & -e & -f \\ g & h & i \end{vmatrix}$$ ← The second row was added to the first row.

$$= -1\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$ ← A common factor of -1 from the second row was taken through the determinant sign.

$$= (-1)(-6) = 6$$

26. $$\begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g+3a & h+3b & i+3c \end{vmatrix} = \begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{vmatrix}$$ ← -3 times the first row was added to the last row.

$$= 2\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$ ← A common factor of 2 from the second row was taken through the determinant sign.

$$= (2)(-6) = -12$$

34. $$\begin{vmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{vmatrix} = \begin{vmatrix} a & b & b & b \\ b-a & a-b & 0 & 0 \\ b-a & 0 & a-b & 0 \\ b-a & 0 & 0 & a-b \end{vmatrix}$$ ← -1 times the first row was added to each of the remaining rows.

$$= \begin{vmatrix} a+b & b & b & b \\ b-a & a-b & 0 & 0 \\ b-a & 0 & a-b & 0 \\ 0 & 0 & 0 & a-b \end{vmatrix}$$ ← The last column was added to the first column.

$$= \begin{vmatrix} a+2b & b & b & b \\ b-a & a-b & 0 & 0 \\ 0 & 0 & a-b & 0 \\ 0 & 0 & 0 & a-b \end{vmatrix}$$ ← The third column was added to the first column.

$$= \begin{vmatrix} a+3b & b & b & b \\ 0 & a-b & 0 & 0 \\ 0 & 0 & a-b & 0 \\ 0 & 0 & 0 & a-b \end{vmatrix}$$ ← The second column was added to the first column.

$$= (a+3b)(a-b)^3$$

2.3 Properties of Determinants; Cramer's Rule

2. $\det(-4A) = \begin{vmatrix} -8 & -8 \\ -20 & 8 \end{vmatrix} = (-8)(8) - (-8)(-20) = -224$

$(-4)^2 \det(A) = 16\begin{vmatrix} 2 & 2 \\ 5 & -2 \end{vmatrix} = 16\big((2)(-2) - (2)(5)\big) = (16)(-14) = -224$

4. $\det(3A) = \begin{vmatrix} 3 & 3 & 3 \\ 0 & 6 & 9 \\ 0 & 3 & -6 \end{vmatrix} = 3\begin{vmatrix} 6 & 9 \\ 3 & -6 \end{vmatrix} = 3\big((6)(-6) - (9)(3)\big) = (3)(-63) = -189$

$3^3\det(A) = 27\begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 1 & -2 \end{vmatrix} = (27)(1)\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = 27\big((2)(-2) - (3)(1)\big) = (27)(-7) = -189$

6. $\det(AB) = \begin{vmatrix} 6 & 15 & 26 \\ 2 & -4 & -3 \\ -2 & 10 & 12 \end{vmatrix} = -66;\ \ \det(BA) = \begin{vmatrix} 5 & 8 & -3 \\ -6 & 14 & 7 \\ 5 & -2 & -5 \end{vmatrix} = -66$

$\det(A+B) = \begin{vmatrix} 1 & 7 & -2 \\ 2 & 1 & 2 \\ -2 & 5 & 1 \end{vmatrix} = -75;\quad \det(A) = 2;\ \ \det(B) = -33;\quad \det(A+B) \neq \det(A) + \det(B)$

8. $\det(A) = -6 \neq 0$ therefore A is invertible by Theorem 2.3.3

10. $\det(A) = 0$ therefore A is not invertible by Theorem 2.3.3

12. $\det(A) = -124 \neq 0$ therefore A is invertible by Theorem 2.3.3

14. $\det(A) = 0$ therefore A is not invertible by Theorem 2.3.3

16. $\det(A) = k^2 - 4 = (k-2)(k+2)$. By Theorem 2.3.3, A is invertible if $k \neq 2$ and $k \neq -2$.

18. $\det(A) = 1 - 4k$. By Theorem 2.3.3, A is invertible if $k \neq \frac{1}{4}$.

20. $\det(A) = -6$ is nonzero, therefore by Theorem 2.3.3, A is invertible.

The cofactors of A are:

$$\begin{array}{lll} C_{11} = -12 & C_{12} = -4 & C_{13} = 6 \\ C_{21} = 0 & C_{22} = -2 & C_{23} = 0 \\ C_{31} = -9 & C_{32} = -4 & C_{33} = 6 \end{array}$$

The matrix of cofactors is

$$\begin{bmatrix} -12 & -4 & 6 \\ 0 & -2 & 0 \\ -9 & -4 & 6 \end{bmatrix}$$

and the adjoint matrix is

$$\text{adj}(A) = \begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix}.$$

From Theorem 2.3.6, we have

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A) = \frac{1}{-6}\begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 0 & \frac{3}{2} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -1 & 0 & -1 \end{bmatrix}.$$

22. $\det(A) = 12$ is nonzero, therefore by Theorem 2.3.3, A is invertible.

The cofactors of A are:

$$\begin{array}{lll} C_{11} = 6 & C_{12} = -48 & C_{13} = 29 \\ C_{21} = 0 & C_{22} = 12 & C_{23} = -6 \\ C_{31} = 0 & C_{32} = 0 & C_{33} = 2 \end{array}$$

The matrix of cofactors is

$$\begin{bmatrix} 6 & -48 & 29 \\ 0 & 12 & -6 \\ 0 & 0 & 2 \end{bmatrix}$$

and the adjoint matrix is

$$\text{adj}(A) = \begin{bmatrix} 6 & 0 & 0 \\ -48 & 12 & 0 \\ 29 & -6 & 2 \end{bmatrix}.$$

From Theorem 2.3.6, we have

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A) = \frac{1}{12}\begin{bmatrix} 6 & 0 & 0 \\ -48 & 12 & 0 \\ 29 & -6 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -4 & 1 & 0 \\ \frac{29}{12} & -\frac{1}{2} & \frac{1}{6} \end{bmatrix}.$$

24. $A = \begin{bmatrix} 7 & -2 \\ 3 & 1 \end{bmatrix}$, $A_1 = \begin{bmatrix} 3 & -2 \\ 5 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 7 & 3 \\ 3 & 5 \end{bmatrix}$; $x_1 = \frac{\det(A_1)}{\det(A)} = \frac{13}{13} = 1$, $x_2 = \frac{\det(A_2)}{\det(A)} = \frac{26}{13} = 2$

26. $A = \begin{bmatrix} 1 & -4 & 1 \\ 4 & -1 & 2 \\ 2 & 2 & -3 \end{bmatrix}$, $A_1 = \begin{bmatrix} 6 & -4 & 1 \\ -1 & -1 & 2 \\ -20 & 2 & -3 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 6 & 1 \\ 4 & -1 & 2 \\ 2 & -20 & -3 \end{bmatrix}$, $A_3 = \begin{bmatrix} 1 & -4 & 6 \\ 4 & -1 & -1 \\ 2 & 2 & -20 \end{bmatrix}$;

$x = \frac{\det(A_1)}{\det(A)} = \frac{144}{-55}$, $y = \frac{\det(A_2)}{\det(A)} = \frac{61}{-55}$, $z = \frac{\det(A_3)}{\det(A)} = \frac{-230}{-55} = \frac{46}{11}$.

28. $A = \begin{bmatrix} -1 & -4 & 2 & 1 \\ 2 & -1 & 7 & 9 \\ -1 & 1 & 3 & 1 \\ 1 & -2 & 1 & -4 \end{bmatrix}$, $A_1 = \begin{bmatrix} -32 & -4 & 2 & 1 \\ 14 & -1 & 7 & 9 \\ 11 & 1 & 3 & 1 \\ -4 & -2 & 1 & -4 \end{bmatrix}$, $A_2 = \begin{bmatrix} -1 & -32 & 2 & 1 \\ 2 & 14 & 7 & 9 \\ -1 & 11 & 3 & 1 \\ 1 & -4 & 1 & -4 \end{bmatrix}$,

$A_3 = \begin{bmatrix} -1 & -4 & -32 & 1 \\ 2 & -1 & 14 & 9 \\ -1 & 1 & 11 & 1 \\ 1 & -2 & -4 & -4 \end{bmatrix}$, $A_4 = \begin{bmatrix} -1 & -4 & 2 & -32 \\ 2 & -1 & 7 & 14 \\ -1 & 1 & 3 & 11 \\ 1 & -2 & 1 & -4 \end{bmatrix}$;

$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-2115}{-423} = 5$, $x_2 = \frac{\det(A_2)}{\det(A)} = \frac{-3384}{-423} = 8$, $x_3 = \frac{\det(A_3)}{\det(A)} = \frac{-1269}{-423} = 3$, $x_4 = \frac{\det(A_4)}{\det(A)} = \frac{423}{-423} = -1$

30. $\det(A) = \cos^2\theta + \sin^2\theta = 1$ is nonzero for all values of θ, therefore by Theorem 2.3.3, A is invertible.

The cofactors of A are:

$$\begin{array}{lll} C_{11} = \cos\theta & C_{12} = \sin\theta & C_{13} = 0 \\ C_{21} = -\sin\theta & C_{22} = \cos\theta & C_{23} = 0 \\ C_{31} = 0 & C_{32} = 0 & C_{33} = \cos^2\theta + \sin^2\theta = 1 \end{array}$$

The matrix of cofactors is

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the adjoint matrix is

$$\operatorname{adj}(A) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

From Theorem 2.3.6, we have

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{1}\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

32. **(a)** $A = \begin{bmatrix} 4 & 1 & 1 & 1 \\ 3 & 7 & -1 & 1 \\ 7 & 3 & -5 & 8 \\ 1 & 1 & 1 & 2 \end{bmatrix}$,

$$A_1 = \begin{bmatrix} 6 & 1 & 1 & 1 \\ 1 & 7 & -1 & 1 \\ -3 & 3 & -5 & 8 \\ 3 & 1 & 1 & 2 \end{bmatrix}, \ A_2 = \begin{bmatrix} 4 & 6 & 1 & 1 \\ 3 & 1 & -1 & 1 \\ 7 & -3 & -5 & 8 \\ 1 & 3 & 1 & 2 \end{bmatrix}, \ A_3 = \begin{bmatrix} 4 & 1 & 6 & 1 \\ 3 & 7 & 1 & 1 \\ 7 & 3 & -3 & 8 \\ 1 & 1 & 3 & 2 \end{bmatrix}, \ A_4 = \begin{bmatrix} 4 & 1 & 1 & 6 \\ 3 & 7 & -1 & 1 \\ 7 & 3 & -5 & -3 \\ 1 & 1 & 1 & 3 \end{bmatrix};$$

$$x = \frac{\det(A_1)}{\det(A)} = \frac{-424}{-424} = 1, \ y = \frac{\det(A_2)}{\det(A)} = \frac{0}{-424} = 0, \ z = \frac{\det(A_3)}{\det(A)} = \frac{-848}{-424} = 2, \ w = \frac{\det(A_4)}{\det(A)} = \frac{0}{-424} = 0$$

(b) The augmented matrix of the system $\begin{bmatrix} 4 & 1 & 1 & 1 & 6 \\ 3 & 7 & -1 & 1 & 1 \\ 7 & 3 & -5 & 8 & -3 \\ 1 & 1 & 1 & 2 & 3 \end{bmatrix}$ has the reduced row echelon form

$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ therefore the system has only one solution: $x = 1, \ y = 0, \ z = 2,$ and $w = 0.$

(c) The method in part (b) requires fewer computations.

36. **(a)** $\det(-A) = \det\big((-1)A\big) = (-1)^4 \det(A) = \det(A) = -2$ (using Formula (1) on p. 106)

(b) $\det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{-2} = -\frac{1}{2}$ (using Theorem 2.3.5)

(c) $\det(2A^T) = 2^4 \det(A^T) = 16 \det(A) = -32$ (using Formula (1) on p. 106 and Theorem 2.2.2)

(d) $\det(A^3) = \det(AAA) = \det(A)\det(A)\det(A) = (-2)^3 = -8$ (using Theorem 2.3.4)

Chapter 2 Supplementary Exercises

2. **(a)** Cofactor expansion along the first row: $\begin{vmatrix} 7 & -1 \\ -2 & -6 \end{vmatrix} = (7)(-6) - (-1)(-2) = -42 - 2 = -44$

(b) $\begin{vmatrix} 7 & -1 \\ -2 & -6 \end{vmatrix} = -\begin{vmatrix} -2 & -6 \\ 7 & -1 \end{vmatrix}$ ← The first and second rows were interchanged.

$= -(-2)\begin{vmatrix} 1 & 3 \\ 7 & -1 \end{vmatrix}$ ← A common factor of -2 from the first row was taken through the determinant sign.

$$= -(-2)\begin{vmatrix} 1 & 3 \\ 0 & -22 \end{vmatrix}$$ ← -7 times the first row was added to the second row

$$= -(-2)(1)(-22) = -44$$ ← Use Theorem 2.1.2.

4. **(a)** Cofactor expansion along the first row:

$$\begin{vmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \\ -7 & -8 & -9 \end{vmatrix} = (-1)\begin{vmatrix} -5 & -6 \\ -8 & -9 \end{vmatrix} - (-2)\begin{vmatrix} -4 & -6 \\ -7 & -9 \end{vmatrix} + (-3)\begin{vmatrix} -4 & -5 \\ -7 & -8 \end{vmatrix}$$

$$= (-1)[(-5)(-9) - (-6)(-8)] - (-2)[(-4)(-9) - (-6)(-7)] + (-3)[(-4)(-8) - (-5)(-7)]$$

$$= (-1)(-3) - (-2)(-6) + (-3)(-3)$$

$$= 3 - 12 + 9$$

$$= 0$$

(b) $$\begin{vmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \\ -7 & -8 & -9 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 2 & 3 \\ -4 & -5 & -6 \\ -7 & -8 & -9 \end{vmatrix}$$ ← A common factor of -1 from the first row was taken through the determinant sign.

$$= (-1)\begin{vmatrix} 1 & 2 & 3 \\ 0 & 3 & 6 \\ 0 & 6 & 12 \end{vmatrix}$$ ← 4 times the first row was added to the second row and 7 times the first row was added to the third row

$$= (-1)\begin{vmatrix} 1 & 2 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{vmatrix}$$ ← -2 times the second row was added to the third row

$$= (-1)(0) = 0$$ ← Use Theorem 2.2.1.

6. **(a)** Cofactor expansion along the second row:

$$\begin{vmatrix} -5 & 1 & 4 \\ 3 & 0 & 2 \\ 1 & -2 & 2 \end{vmatrix} = -3\begin{vmatrix} 1 & 4 \\ -2 & 2 \end{vmatrix} + 0 - 2\begin{vmatrix} -5 & 1 \\ 1 & -2 \end{vmatrix}$$

$$= -3[(1)(2) - (4)(-2)] - 2[(-5)(-2) - (1)(1)]$$

$$= (-3)(10) - 2(9)$$

$$= -30 - 18$$

$$= -48$$

(b)

$$\begin{vmatrix} -5 & 1 & 4 \\ 3 & 0 & 2 \\ 1 & -2 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & -2 & 2 \\ 3 & 0 & 2 \\ -5 & 1 & 4 \end{vmatrix}$$

← The first and third rows were interchanged.

$$= -\begin{vmatrix} 1 & -2 & 2 \\ 0 & 6 & -4 \\ 0 & -9 & 14 \end{vmatrix}$$

← -3 times the first row was added to the second row and 5 times the first row was added to the third row

$$= -6\begin{vmatrix} 1 & -2 & 2 \\ 0 & 1 & -\frac{2}{3} \\ 0 & -9 & 14 \end{vmatrix}$$

← A common factor of 6 from the second row was taken through the determinant sign.

$$= -6\begin{vmatrix} 1 & -2 & 2 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 8 \end{vmatrix}$$

← 9 times the second row was added to the third row.

$$= -6(1)(1)(8) = -48$$

← Use Theorem 2.1.2.

8. **(a)** We perform cofactor expansions along the first row in the 4x4 determinant, as well as in each of the 3x3 determinants:

$$\begin{vmatrix} -1 & -2 & -3 & -4 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ -4 & -3 & -2 & -1 \end{vmatrix}$$

$$= -1\begin{vmatrix} 3 & 2 & 1 \\ 2 & 3 & 4 \\ -3 & -2 & -1 \end{vmatrix} - (-2)\begin{vmatrix} 4 & 2 & 1 \\ 1 & 3 & 4 \\ -4 & -2 & -1 \end{vmatrix} + (-3)\begin{vmatrix} 4 & 3 & 1 \\ 1 & 2 & 4 \\ -4 & -3 & -1 \end{vmatrix} - (-4)\begin{vmatrix} 4 & 3 & 2 \\ 1 & 2 & 3 \\ -4 & -3 & -2 \end{vmatrix}$$

$$= -1\left(3\begin{vmatrix} 3 & 4 \\ -2 & -1 \end{vmatrix} - 2\begin{vmatrix} 2 & 4 \\ -3 & -1 \end{vmatrix} + 1\begin{vmatrix} 2 & 3 \\ -3 & -2 \end{vmatrix}\right) + 2\left(4\begin{vmatrix} 3 & 4 \\ -2 & -1 \end{vmatrix} - 2\begin{vmatrix} 1 & 4 \\ -4 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & 3 \\ -4 & -2 \end{vmatrix}\right)$$

$$-3\left(4\begin{vmatrix} 2 & 4 \\ -3 & -1 \end{vmatrix} - 3\begin{vmatrix} 1 & 4 \\ -4 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ -4 & -3 \end{vmatrix}\right) + 4\left(4\begin{vmatrix} 2 & 3 \\ -3 & -2 \end{vmatrix} - 3\begin{vmatrix} 1 & 3 \\ -4 & -2 \end{vmatrix} + 2\begin{vmatrix} 1 & 2 \\ -4 & -3 \end{vmatrix}\right)$$

$$= -\big((3)(5) - (2)(10) + 5\big) + (2)((4)(5) - 2\big((4)(5) - (2)(15) + 10\big)$$

$$-3\big((4)(10) - (3)(15) + 5\big) + 4\big((4)(5) - (3)(10) + (2)(5)\big)$$

$$= 0 + 0 + 0 + 0$$

$$= 0$$

(b)

$$\begin{vmatrix} -1 & -2 & -3 & -4 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ -4 & -3 & -2 & -1 \end{vmatrix} = \begin{vmatrix} -1 & -2 & -3 & -4 \\ 4 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ -4 & -3 & -2 & -1 \end{vmatrix}$$

← The first row was added to the third row.

$$= 0$$

← Use Theorem 2.2.1.

10. **(a)** e.g. $\begin{vmatrix} 4 & 0 & 3 & 6 \\ 8 & 0 & 5 & 0 \\ 7 & 3 & 7 & 10 \\ 13 & 0 & 10 & 0 \end{vmatrix} = -3\begin{vmatrix} 4 & 3 & 6 \\ 8 & 5 & 0 \\ 13 & 10 & 0 \end{vmatrix} = -18\begin{vmatrix} 8 & 5 \\ 13 & 10 \end{vmatrix} = (-18)(15) = -270$ was easy to calculate by cofactor expansions (first, we expanded along the second column, then along the third column), but would be more difficult to calculate using elementary row operations.

(b) e.g., $\begin{vmatrix} -1 & -2 & -3 & -4 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ -4 & -3 & -2 & -1 \end{vmatrix}$ of Exercise 8 was easy to calculate using elementary row operations, but more difficult using cofactor expansion.

12. In Exercise 5: $\begin{vmatrix} 3 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 4 & 2 \end{vmatrix} = -10 \neq 0$ therefore the matrix is invertible.

In Exercise 6: $\begin{vmatrix} -5 & 1 & 4 \\ 3 & 0 & 2 \\ 1 & -2 & 2 \end{vmatrix} = -48 \neq 0$ therefore the matrix is invertible.

In Exercise 7: $\begin{vmatrix} 3 & 6 & 0 & 1 \\ -2 & 3 & 1 & 4 \\ 1 & 0 & -1 & 1 \\ -9 & 2 & -2 & 2 \end{vmatrix} = 329 \neq 0$ therefore the matrix is invertible.

In Exercise 8: $\begin{vmatrix} -1 & -2 & -3 & -4 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ -4 & -3 & -2 & -1 \end{vmatrix} = 0$ therefore the matrix is not invertible.

14. $\begin{vmatrix} 3 & -4 & a \\ a^2 & 1 & 2 \\ 2 & a-1 & 4 \end{vmatrix} = \begin{vmatrix} 4a^2+3 & 0 & 8+a \\ a^2 & 1 & 2 \\ -a^3+a^2+2 & 0 & -2a+6 \end{vmatrix}$ ← 4 times the second row was added to the first row and $1-a$ times the second row was added to the last row.

$= -0 + 1\begin{vmatrix} 4a^2+3 & 8+a \\ -a^3+a^2+2 & -2a+6 \end{vmatrix} - 0$ ← Cofactor expansion along the second column.

$= (4a^2+3)(-2a+6) - (8+a)(-a^3+a^2+2)$

$= a^4 - a^3 + 16a^2 - 8a + 2$

16. $\begin{vmatrix} x & -1 \\ 3 & 1-x \end{vmatrix} = x(1-x) - (-1)(3) = -x^2 + x + 3;$

Adding -2 times the first row to the second row, then performing cofactor expansion along the second row yields $\begin{vmatrix} 1 & 0 & -3 \\ 2 & x & -6 \\ 1 & 3 & x-5 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -3 \\ 0 & x & 0 \\ 1 & 3 & x-5 \end{vmatrix} = x\begin{vmatrix} 1 & -3 \\ 1 & x-5 \end{vmatrix} = x(x-5+3) = x^2 - 2x$

Solve the equation

$$-x^2 + x + 3 = x^2 - 2x$$
$$-2x^2 + 3x + 3 = 0$$

From quadratic formula $x = \frac{-3+\sqrt{9+24}}{-4} = \frac{3-\sqrt{33}}{4}$ or $x = \frac{-3-\sqrt{9+24}}{-4} = \frac{3+\sqrt{33}}{4}$.

18. $\begin{vmatrix} 7 & -1 \\ -2 & -6 \end{vmatrix} = -44$ is nonzero, therefore by Theorem 2.3.3, the matrix $A = \begin{bmatrix} 7 & -1 \\ -2 & -6 \end{bmatrix}$ is invertible.

The cofactors are:

$$\begin{matrix} C_{11} = -6 & C_{12} = 2 \\ C_{21} = 1 & C_{22} = 7 \end{matrix}$$

The matrix of cofactors is

$$\begin{bmatrix} -6 & 2 \\ 1 & 7 \end{bmatrix}$$

and the adjoint matrix is

$$\text{adj}(A) = \begin{bmatrix} -6 & 1 \\ 2 & 7 \end{bmatrix}.$$

From Theorem 2.3.6, we have

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A) = \frac{1}{-44}\begin{bmatrix} -6 & 1 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} \frac{3}{22} & -\frac{1}{44} \\ -\frac{1}{22} & -\frac{7}{44} \end{bmatrix}.$$

20. $\begin{vmatrix} -1 & -2 & -3 \\ -4 & -5 & -6 \\ -7 & -8 & -9 \end{vmatrix} = 0$ therefore by Theorem 2.3.3, the matrix is not invertible

22. $\begin{vmatrix} -5 & 1 & 4 \\ 3 & 0 & 2 \\ 1 & -2 & 2 \end{vmatrix} = -48$ is nonzero, therefore by Theorem 2.3.3, $A = \begin{bmatrix} -5 & 1 & 4 \\ 3 & 0 & 2 \\ 1 & -2 & 2 \end{bmatrix}$ is invertible.

The cofactors of A are:

$$\begin{array}{lll} C_{11} = 4 & C_{12} = -4 & C_{13} = -6 \\ C_{21} = -10 & C_{22} = -14 & C_{23} = -9 \\ C_{31} = 2 & C_{32} = 22 & C_{33} = -3 \end{array}$$

The matrix of cofactors is

$$\begin{bmatrix} 4 & -4 & -6 \\ -10 & -14 & -9 \\ 2 & 22 & -3 \end{bmatrix}$$

and the adjoint matrix is

$$\text{adj}(A) = \begin{bmatrix} 4 & -10 & 2 \\ -4 & -14 & 22 \\ -6 & -9 & -3 \end{bmatrix}.$$

From Theorem 2.3.6, we have

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A) = \frac{1}{-48}\begin{bmatrix} 4 & -10 & 2 \\ -4 & -14 & 22 \\ -6 & -9 & -3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{12} & \frac{5}{24} & -\frac{1}{24} \\ \frac{1}{12} & \frac{7}{24} & -\frac{11}{24} \\ \frac{1}{8} & \frac{3}{16} & \frac{1}{16} \end{bmatrix}.$$

24. $\begin{vmatrix} -1 & -2 & -3 & -4 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ -4 & -3 & -2 & -1 \end{vmatrix} = 0$ therefore by Theorem 2.3.3, the matrix is not invertible

26. $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $A_1 = \begin{bmatrix} x & -\sin\theta \\ y & \cos\theta \end{bmatrix}$, $A_2 = \begin{bmatrix} \cos\theta & x \\ \sin\theta & y \end{bmatrix}$;

$x' = \frac{\det(A_1)}{\det(A)} = \frac{x\cos\theta + y\sin\theta}{\cos^2\theta + \sin^2\theta} = x\cos\theta + y\sin\theta$, $y' = \frac{\det(A_2)}{\det(A)} = \frac{y\cos\theta - x\sin\theta}{\cos^2\theta + \sin^2\theta} = y\cos\theta - x\sin\theta$

28. According to the arrow technique (see Example 7 in Section 2.1), the determinant of a 3×3 matrix can be expressed as a sum of six terms:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

If each entry of A is either 0 or 1, then each of the terms must be either 0 or ± 1. The largest value 3 would result from the terms $1 + 1 + 1 - 0 - 0 - 0$, however, this is not possible since the first three terms all equal 1 would require that all nine matrix entries be equal 1, making the determinant 0.

The largest value of determinant that is actually attainable is 2, e.g., let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

34. (b) $\frac{1}{2}\begin{vmatrix} 3 & 3 & 1 \\ 4 & 0 & 1 \\ -2 & -1 & 1 \end{vmatrix} = -\frac{19}{2}$ is the negative of the area of the triangle because it is being traced clockwise; (reversing the order of the points would change the orientation to counterclockwise, and thereby result in the positive area: $\frac{1}{2}\begin{vmatrix} -2 & -1 & 1 \\ 4 & 0 & 1 \\ 3 & 3 & 1 \end{vmatrix} = \frac{19}{2}$)