Instructor Solutions Manual
to accompany
Single Variable Calculus
by
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Contents

Chapter 1

Basics

1.1 Number Systems

Problems for Practice

1. $2.13 = 213/100$

2. $0.00034 = 34/100000$

3. Let $x = 0.232323\ldots$. Then $100x = 23.232323\ldots$. Subtract to obtain $99x = 23$, implying that $x = 23/99$.

4. Let $x = 0.222\ldots$. Then $10x = 2.222\ldots$. Subtract to obtain $9x = 2$, implying that $x = 2/9$.

5. Let $x = 5.001001001\ldots$. Then $1000x = 5001.001001001\ldots$. Subtract to obtain $999x = 4996$, implying that $x = 4996/999$.

6. Let $x = 15.7231231231\ldots$. Then $10x = 157.231231231\ldots$, and $10000x = 157231.231231\ldots$. Subtract to obtain $9990x = 157074$, implying that $x = 157074/9990$.

7. Divide 40 into 1.00000 until it terminates to yield a repeating block of zeros: $1/40 = 0.025\overline{0} = 0.025$.

8. Divide 8 into 25.00000 until it terminates to yield a repeating block of zeros: $25/8 = 3.125\overline{0} = 3.125$.

9. Divide 25 into 18.00000 until it terminates to yield a repeating block of zeros: $18/25 = 0.72\overline{0} = 0.72$.

10. Divide 3 into 5.00000 until it repeats to yield a repeating block of the digit 6: $5/3 = 1.\overline{6}$.

11. Divide 7 into 2.0000000 until the initial remainder (2) repeats yielding a long repeating block: $2/7 = 0.\overline{285714}$.

12. Divide 7 into 31.0000000 until the second remainder (1) repeats yielding a long repeating block: $31/7 = 2.2\overline{142857}$.

13. $\{x : 1 \leq x \leq 3\} = [\,1, 3\,]$

14. The inequality $|x-2| < 5$ is equivalent to $-5 < x-2 < 5$, or $-3 < x < 7$. Therefore, $\{x : |x-2| < 5\} = (\,-3, 7\,)$.

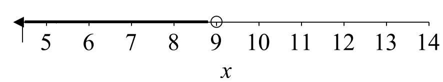
15. $\{t : t > 1\} = (\,1, \infty\,)$

16. A number u makes $|u-4| \geq 6$ true when it is at least 6 units from the number 4 on the number line. Therefore, u must be at 10 (or to the right) or at -2 (or to the left). That is, $\{u : |h-4| \geq 6\} = (\,-\infty, -2\,] \cup [\,10, \infty\,)$.

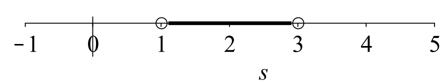
17. The number y satisfies the inequality $|y+4| \leq 10$ when it is 10 units or less from the number -4 on the number line. That is, it must be at, or to the left of, 6 and at, or to the right of, -14: $\{y : |y+4| \leq 10\} = [\,-14, 6\,]$.

18. $|s-2|$ is greater than 8 when s is greater than 8 units from the number 2 on the number line. Thus s must be to the right of 10 or to the left of -6: $\{s : |s-2| < 8\} = (-\infty, -6) \cup (10, \infty)$.

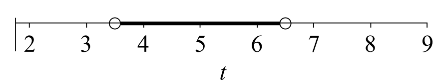
19. Subtract x from each side, and then add 5 to each side of the given inequality to obtain $x < 9$. See the picture.



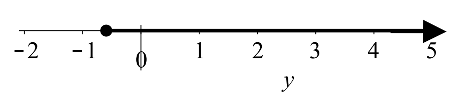
20. These are all numbers s that are less than one unit from 2. See the picture.



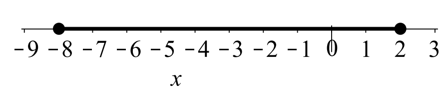
21. Take the square root on both sides. Order is preserved and we get $|t-5| < 3/2$. That is, all numbers that are less than 3/2 units from 5. See the picture.



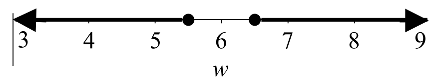
22. Subtract $2y$ from both sides of the inequality, then subtract 4 from both sides to get $5y \geq -3$ or $y \geq -3/5$. See the picture.



23. Divide both sides by 3 to obtain the inequality $|x+3| \leq 5$. This is all numbers that are less than or equal to 5 units from -3. See the picture.



24. Divide both sides by 2 to obtain the inequality $|w-6| \geq 1/2$. This is all numbers that are at least 1/2 unit from 6. See the picture.



25. The midpoint of the interval is $c = 1$ and its width is 4, so
$$[-1, 3] = \{x : |x - 1| \leq 2\}.$$

26. The midpoint of the interval is $c = \frac{1}{2}(3 + 4\sqrt{2})$ and its width is $4\sqrt{2} - 3$, so $[3, 4\sqrt{2}] = \{x : |x - 3/2 - 2\sqrt{2}\,)\,| \leq 2\sqrt{2} - 3/2\}$.

27. The midpoint of the interval is $c = 1$ and its width is $2\pi + 2$, so
$$(-\pi, \pi + 2) = \{x : |x - 1| < \pi + 1\}.$$

28. The midpoint of the interval is $c = \pi - \sqrt{2}/2$ and its width is $\sqrt{2}$, so
$$(\pi - \sqrt{2}, \pi) = \{x : |x - \pi + \sqrt{2}/2| < \sqrt{2}/2\}.$$

Further Theory and Practice

29. Rational numbers x and y are of the form $x = m/n$ and $y = p/q$ where m, n, p, and q are integers. Therefore,

$$x + y = m/n + p/q = (m \cdot q + p \cdot n)/(p \cdot q) \quad \text{and} \quad x \cdot y = (m \cdot n)/(p \cdot q)\,.$$

Both x and y are rational because the products and sums of integers are integers.

30. $\sqrt{8}$ and $\sqrt{2}$ are irrational. Their product, $\sqrt{8} \cdot \sqrt{2} = \sqrt{16} = 4$, is rational.

The numbers $\sqrt{2} + 1$ and $3 - \sqrt{2}$ are irrational. Their sum, $(\sqrt{2} + 1) + (3 - \sqrt{2}\,) = 4$ is rational.

31. The numbers 22/7 and 3.14 are both rational, π is irrational.

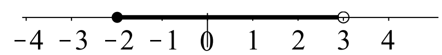
32. Since $a = \pi - 3.14 \approx 0.00159$ and $b = 22/7 - \pi \approx 0.00126$, the smallest closed interval, centered at π, and containing both 22/7 and 3.14, has radius $r = a = \pi - 3.14$. The interval is $\{x : |x - \pi| \leq \pi - 3.14\}$. As the difference of irrational π and rational 3.14, r's decimal expansion does not terminate or repeat. It is irrational.

33. One percent of the correct mass is $\epsilon = 0.345$. With $c = 34.5$, the interval is $\{x : |x - 34.5| \leq 0.345\} = [34.155, 34.845]$.

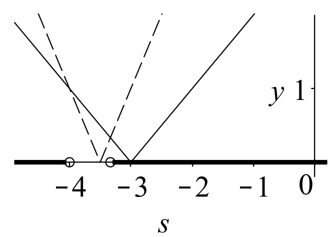
34. Let w and h denote the width and height of the finished door. Ignoring expansion, the maximum width (in millimeters, mm) is $895 - 2 = 893$. This should be reduced by 0.2% in case of expansion so $w \leq 893 \cdot 0.998 = 891.214$ mm. Similarly, $h \leq (1485 - 2) \cdot 0.998 = 1480.034$ mm.

The minimum door width is $895 - 14 = 881$ mm so $881 \leq w \leq 891$. The minimum door height is $1485 - 14 = 1471$ mm so $1471 \leq h \leq 1480$.

35. This set is the intersection of the intervals $[-2, \infty)$ and $(-3, 3)$: $\{x : -2 \leq x < 3\}$. See the picture.



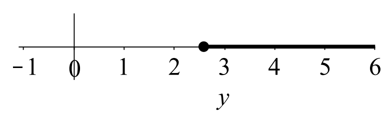
36. Plot $y = |s + 3|$ and $y = |2s + 7|$ to see that $|s+3|$ is less than $|2s+7|$ if $s < -4$ or if $s > -10/3$. These are the intersection of lines $y = -(s + 3)$ and $y = -(2s + 7)$ and the intersection of lines $y = -(s + 3)$ and $y = 2s + 7$. See the picture.



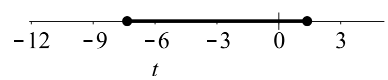
37. This is the intersection of the semi-infinite intervals $\{y : -4 - \sqrt{7} < 2y\}$ and $\{y : 4 - \sqrt{2} \leq y\}$. Since

$$-2 - \sqrt{7}/2 < 4 - \sqrt{2},$$

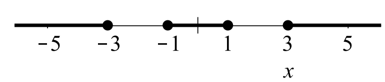
it is the interval $[4 - \sqrt{2}, \infty)$. See the picture.



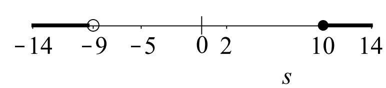
38. The inequality is equivalent to $-10 \leq t^2 + 6t \leq 10$. Add 9 to all three parts to obtain the inequality $-1 \leq (t + 3)^2 \leq 19$ which, in turn, is equivalent to $|t + 3| \leq \sqrt{19}$. This is all numbers within $\sqrt{19}$ units of -3. See the picture.



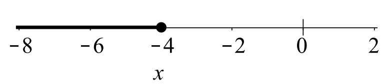
39. The distance from x^2 to 5 must be 4 or more. Therefore, one possibility is $x^2 = 9$ or more and another is $x^2 = 1$ or less. That is, $|x| \geq 3$ or $|x| \leq 1$. See the picture.



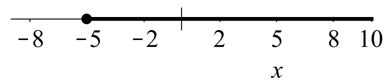
40. If the number s is greater than -5, then it must be greater than -1, but also greater than or equal to 10. This yields the semi-infinite interval $[10, \infty)$. If $s < -5$, then must be less than -9 and less than or equal to -6. This yields the semi-infinite interval $(-\infty, -9)$. See the picture.



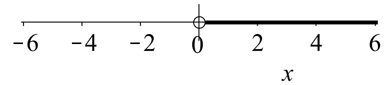
41. This is the intersection of two intervals. One of them is $\{x : -4 \geq x\}$ obtained by subtracting $x+5$ from the left and middle parts of the defining inequalities. The other is $\{x : -3 > x\}$ obtained by subtracting $2x+8$ from the middle and right parts of the defining inequalities. Intersecting yields just one interval: $(-\infty, -4]$. See the picture.



42. This is the union of two intervals I and J. $I = \{t : |t-4| < |t-2|\}$. That is, t is closer to 4 than it is to 2. Equivalently, $I = (3, \infty)$. The other is $J = [-5, 3]$ and $I \cup J = [-5, \infty)$. See the picture.



43. x cannot be 0. If $x > 0$, then the inequality is equivalent to $|x+1| > x-1$, which is true for all such x. If $x < 0$, then the inequality is equivalent to $|-x-1| > -x+1$ or $|x+1| > 1-x$ which is true for no such x. See the picture.



44. $\{x : |x-2| \le 4\} = [-2, 6]$

45. $\{x : |x-3| = |x+9|\} = \{-3\}$

46. $\{x : 2 < x^2 < 10\} = \{x : \sqrt{2} < |x| < \sqrt{10}|\}$
$$= (-\sqrt{10}, -\sqrt{2}) \cup (\sqrt{2}, \sqrt{10})$$

47. $\{x : |x-3| < 2 \text{ and } x^2 \le 8\} = (1, 5) \cap [-2\sqrt{2}, 2\sqrt{2}] = (1, 2\sqrt{2}]$

48. $\{x : x < -1 \text{ and } x+5 < -(x+1)\} = \{x : x < -3\}$

49. $\{s : s < 4 \text{ and } |2s+9| < 4-s\} = \{s : s < 4 \text{ and } s-4 < 2s+9 < 4-s\}$
$$= \{s : s < 4 \text{ and } -13 < s < -5-2s\} = \{s : -13 < s < -5/3\}$$

50. $\{t : 0 < t^2-2t\} = \{t : 1 < t^2-2t+1\} = \{t : 1 < (t-1)^2\} = \{t : 1 < |t-1|\}$

51. $\{w : w/(w+1) < 0\} = \{w : -1 < w < 0\} = \{w : |w+1/2| < 1/2\}$

52. Since $p(x) = (x-r_1)(x-r_2)$, $\{x : p(x) \le 0\} = \{x : r_1 \le x \le r_2\}$
$$= [r_1, r_2].$$

53. When a and b are nonpositive so is $a+b$ and
$$|a+b| = -(a+b) = -a + (-b) = |a| + |b| \le |a| + |b|.$$

54. Since $a = (a+b)+(-b)$, $|a| = |(a+b)+(-b)|$. Using the triangle inequality on the right side: $|a| \le |a+b| + |-b|$. But $|-b| = |b|$, so subtract $|b|$ from both sides of the previous inequality to obtain $|a| - |b| \le |a+b|$.

55. If z were the smallest positive number, then $0 < z < 1$. Multiply all three parts by z to conclude that $0 < z^2 < z$ and z^2 would be smaller than z, a contradiction.

56. Since $x = 1 - 0.999\ldots$, given any positive integer n,
$$10^n \cdot x = 10^n - \underbrace{99\cdots9}_{n \text{ 9s}}.999\ldots < 10^n - \underbrace{99\cdots9}_{n \text{ 9s}}.9 = 0.1 < 1.$$
Therefore, $0 \le x < 10^{-n}$. Consequently, given any positive number b, $0 \le x < b$ (this is because there is an integer n such that $10^n > 1/b$, so $10^{-n} < b$). It follows that $x = 0$ for if not, then $x > 0$, and it would be the case that $0 < x < x$. Because $x = 0$, $1 = 0.999\ldots$.

57. d. Since $a^2 = 2b^2$, a^2 is an even integer. This implies that a is also even (the square of an odd integer must be odd). Therefore, 2 divides a evenly. That is, $a = 2\alpha$ for some integer α.

e. Substitute for a in the equation $a^2 = 2b^2$ to conclude that $4\alpha^2 = 2b^2$. This implies that $b^2 = 2\alpha^2$.

f. Since the integer b^2 is even, b is also even and 2 divides b with no remainder.

g. From d we conclude that a is even. From f we conclude that b even. This contradicts the fact that a and b are integers with no common factors.

58. While it is true that $(x-3)^2 = (x-\pi)^2$, we may only conclude that $|x-3| = |x-\pi|$. This leaves *two* possibilities: either $x-3 = x-\pi$, or $x-3 = -(x-\pi)$. Since the first possibility is obviously false, we conclude that $x-3 = -(x-\pi)$. That is, $x = (\pi+3)/2$, which is in agreement with the original definition of x.

CALCULATOR/COMPUTER EXERCISES

59. Add and subtract 0.0005 to $x = 0.449$. We conclude that y must lie in the open interval $(0.4485, 0.4495)$.

60. Add and subtract 0.005 to $x = 24.00$. We conclude that y must lie in the open interval $(23.995, 24.005)$.

61. Add and subtract $5 \cdot 10^{-4}$ to $x = 0.999 \cdot 10^{-5}$. We conclude that y must lie in the open interval $(-0.49001 \cdot 10^{-3}, 0.50999 \cdot 10^{-3})$.

62. Add and subtract $5 \cdot 10^{-6}$ to $x = 0.213462 \cdot 10^{-1}$. We conclude that y must lie in the open interval $(0.213412 \cdot 10^{-1}, 0.213512 \cdot 10^{-1})$.

63. The number $y = 4.001$ has the property that $|x-y| = 0.005$ so it agrees with x to two decimal places.

64. Rounding to three we obtain 0.445. Rounding to two we get 0.44, and rounding to one decimal we have 0.4. Rounding *successively* we get 0.445, then 0.45, and finally 0.5.

This might be characterized as "successive rounding is less accurate".

65. Using *Maple* with 10 significant digits, the product is 0. Using 20 significant digits, the product is 1.036 888 824, and using 30 significant digits, the product is 0.999 999 999 9.

66. Let $a = 0.55004$ and $b = 0.54995$.

67. The relative errors for $x = 10^n$, $n = 6, 11, 16, 21$ are on the left. The relative errors for $x = -10^n$ are on the right.

x	**Relative Error**
10^6	0.99999
10^{11}	0.090909
10^{16}	0.9999×10^{-8}
10^{21}	1.000000×10^{-11}

x	**Relative Error**
-10^6	1.0001
-10^{11}	0.11111
-10^{16}	0.1000001×10^{-5}
-10^{21}	1.000000×10^{-11}

68. The relative errors for $x = 10^n$, $n = 5, 8, 11, 14$ are on the left. The relative errors for $x = -10^n$ are on the right.

x	**Relative Error**
10^5	0.10988
10^8	0.0001234
10^{11}	0.12345×10^{-8}
10^{14}	0.12345×10^{-11}

x	**Relative Error**
-10^5	0.14084
-10^8	0.12347×10^{-3}
-10^{11}	0.12345×10^{-8}
-10^{14}	0.12345×10^{-11}

69. The relative errors for $x = 10^n$, $n = 5, 8, 11, 14$ are on the left. The relative errors for $x = -10^n$ are on the right.

x	**Relative Error**
10^5	0.5
10^8	0.0003334
10^{11}	0.33333×10^{-8}
10^{14}	0.33333×10^{-11}

x	**Relative Error**
-10^5	0.25
-10^8	0.000333
-10^{11}	0.33333×10^{-8}
-10^{14}	0.33333×10^{-11}

70. The relative errors for $x = 10^n$, $n = 5, 8, 11, 14$ are on the left. The relative errors for $x = -10^n$ are on the right.

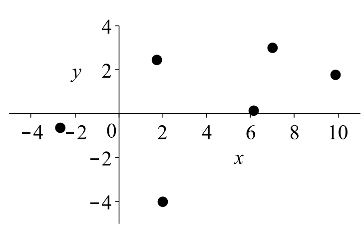
x	**Relative Error**
10^5	0.009901
10^8	0.99999×10^{-5}
10^{11}	0.99999×10^{-10}
10^{14}	1.00000×10^{-11}

x	**Relative Error**
-10^5	0.009901
-10^8	0.99999×10^{-5}
-10^{11}	0.99999×10^{-10}
-10^{14}	1.00000×10^{-11}

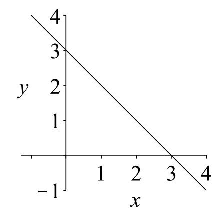
1.2 Planar Coordinates and Graphing in the Plone

Problems for Practice

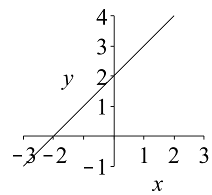
1. The six points are plotted on the right.



2. This is a straight line. See the plot on the right.



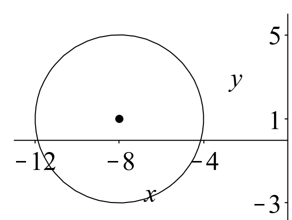
3. This is a straight line. See the plot on the right.



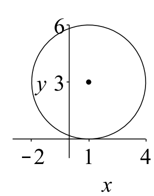
4. The point $(1/2, 9/2)$ is farthest from the origin and $(2, 1)$ is nearest since $\sqrt{5}$ is less than 3. The point $(4, -1)$ is farthest from $(-5, 6)$. The point that is nearest to $(10, 7)$ is $(1/2, 9/2)$.

5. The distance from A to B is $|\overline{AB}| = \sqrt{(2+4)^2 + (3-7)^2} = 2\sqrt{13}$. The distance from A to C is $|\overline{AC}| = \sqrt{(2+5)^2 + (3+6)^2} = \sqrt{130}$. The distance from B to C is $|\overline{BC}| = \sqrt{(-4+5)^2 + (7+6)^2} = \sqrt{170}$.

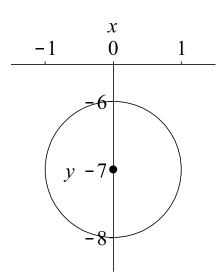
6. The center is at $(-8, 1)$ and the radius is 4. See the plot on the right.



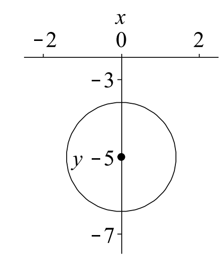
7. The center is at $(1, 3)$ and the radius is 3. See the plot on the right.



8. The center is at $(0, -7)$ and the radius is 1. See the plot on the right.



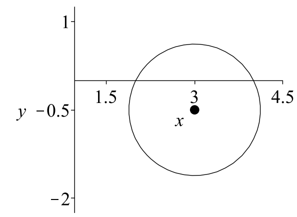
9. The center is at $(0,-5)$ and the radius is $\sqrt{2}$. See the plot on the right.



10. Complete the square in y:

$$(x-3)^2 + \left(y^2 + y + \tfrac{1}{4}\right) = 1 + \tfrac{1}{4}$$

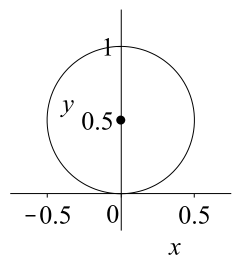
or $(x-3)^2 + (y+1/2)^2 = 5/4$. The center is at $(3,-1/2)$ and the radius is $\sqrt{5}/2$. See the plot on the right.



11. Complete the square in y:

$$x^2 + \left(y^2 - y + \tfrac{1}{4}\right) = 0 + \tfrac{1}{4}$$

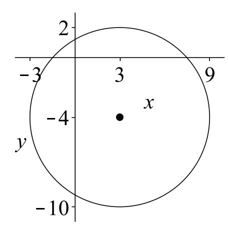
or $x^2+(y-1/2)^2 = 1/4$. The center is at $(0,1/2)$ and the radius is $1/2$. See the plot on the right.



12. Complete the square in x and y:

$$(x^2-6x+9)+(y^2+8y+16) = 11+9+16$$

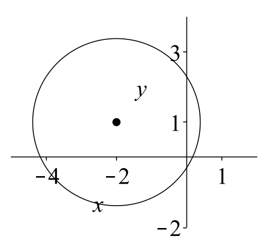
or $(x-3)^2+(y+4)^2 = 36$. The center is at $(3,-4)$ and the radius is 6. See the plot on the right.



13. Complete the square in x and y:

$$3(x^2+4x+4)+3(y^2-2y+1) = 2+12+3$$

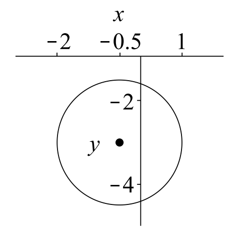
or $(x+2)^2 + (y-1)^2 = 17/3$. The center is at $(-2,1)$ and the radius is $\sqrt{17/3}$. See the plot on the right.



14. Complete the square in x and y:

    $(x^2{+}x{+}1/4){+}(y^2{+}6y{+}9) = -7{+}9{+}1/4$

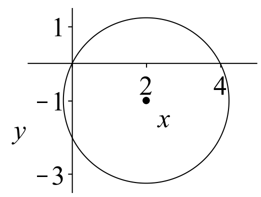
    or $(x+1/2)^2 + (y+3)^2 = 9/4$. The center is at $(-1/2, -3)$ and the radius is $3/2$. See the plot on the right.



15. Complete the square in x and y:

    $4(x^2{-}4x{+}4){+}4(y^2{+}2y{+}1) = 0{+}16{+}4$

    or $(x-2)^2+(y+1)^2 = 5$. The center is at $(2, -1)$ and the radius is $\sqrt{5}$. See the plot on the right.



16. Center at (0,0), radius 2: $x^2 + y^2 = 4$.
17. Center at $(-3, 5)$, radius 6: $(x+3)^2 + (y-5)^2 = 36$.
18. Center at $(3, 0)$, radius 4: $(x-3)^2 + y^2 = 16$.
19. Center at $(-4, \pi)$, radius 5: $(x+4)^2 + (y-\pi)^2 = 25$.
20. Center at $(0, -1/4)$, radius 1/4: $x^2 + (y+1/4)^2 = 1/16$.
21. The vertex is at $(0, -3)$ and the axis of symmetry is the line $x = 0$ (the y-axis).
22. The vertex is at $(3, 4)$ and the axis of symmetry is the line $x = 3$.
23. Since $-B/(2A) = -2/(-2) = 1$, the vertex is at $(1, 1)$ and the axis of symmetry is the line $x = 1$.
24. Since $-B/(2A) = -(-6)/6 = 1$, the vertex is at $(1, -2)$ and the axis of symmetry is the line $x = 1$.
25. Since $y = -x^2 - 6x - 9/2$, $-B/(2A) = -(-6)/(-2) = -3$. The vertex is at $(-3, 9/2)$ and the axis of symmetry is the line $x = -3$.
26. Since $y = (1/3)x^2 - (1/3)x - 1/3$, $-B/(2A) = -(-1/3)/(2/3) = 1/2$. The vertex is at $(1/2, -5/12)$ and the axis of symmetry is the line $x = 1/2$.
27. This is an ellipse. Its standard form equation is $\frac{x^2}{(1/2)^2} + \frac{y^2}{1^2} = 1$ and its center is at the origin, (0,0).
28. This is a hyperbola. Its standard form equation is $\frac{y^2}{1^2} - \frac{(x-5)^2}{1^2} = 1$ and its center is at the point (5,0).
29. This is an ellipse. Complete the square in x: $(x^2 + x + 1/4) + 9y^2 = 15/4 + 1/4$ or $(x+1/2)^2 + 9y^2 = 4$. Consequently, its standard form

equation is $\frac{(x-(-1/2))^2}{2^2} + \frac{y^2}{(2/3)^2} = 1$, so its center is at the point $(-1/2, 0)$.

30. This is a hyperbola. Complete the square in y: $9x^2 - (y^2 + y + 1/4) = 1/2 - 1/4$ or $9x^2 - (y + 1/2)^2 = 1/4$. Consequently, its standard form equation is $\frac{x^2}{(1/6)^2} - \frac{(y-(-1/2))^2}{(1/2)^2} = 1$, so its center is at the point $(0, -1/2)$.

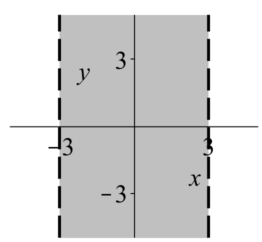
31. This is an ellipse. Complete the square in x and y: $(x^2 + 2x + 1) + 4(y^2 + 6y + 9) = 12 + 1 + 36$ or $(x + 1)^2 + 4(y + 3)^2 = 49$. Consequently, its standard form equation is $\frac{(x-(-1))^2}{7^2} + \frac{(y-(-3))^2}{(7/2)^2} = 1$, so its center is at the point $(-1, -3)$.

32. This is a hyperbola. Complete the square in x and y: $(x^2 - 10x + 25) - (y^2 + 8y + 16) = 0 + 25 - 16$ or $(x - 5)^2 - (y + 4)^2 = 9$. Consequently, its standard form equation is $\frac{(x-5)^2}{3^2} - \frac{(y-(-4))^2}{3^2} = 1$, so its center is at the point $(5, -4)$.

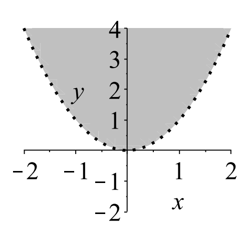
33. This is a hyperbola. Complete the square in y: $2x^2 - 3(y^2 - 2y + 1) = 103 - 3$ or $2x^2 - 3(y-1)^2 = 100$. Consequently, its standard form equation is $\frac{x^2}{(\sqrt{50})^2} - \frac{(y-1)^2}{(10/\sqrt{3})^2} = 1$, so its center is at the point $(0, 1)$.

34. This is an ellipse. Complete the square in x and y: $(1/4)(x^2 + 4x + 4) + (y^2 + 6y + 9) = 6 + 1 + 9$ or $(1/4)(x + 2)^2 + (y + 3)^2 = 16$. Consequently, its standard form equation is $\frac{(x-(-2))^2}{8^2} + \frac{(y-(-3))^2}{4^2} = 1$, so its center is at the point $(-2, -3)$.

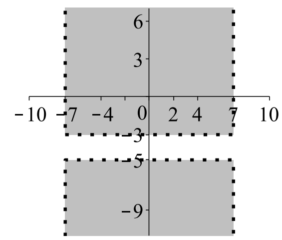
35. The region is all points *between* the vertical lines $x = -3$ and $x = 3$. It is sketched on the right.



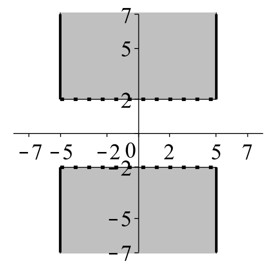
36. The region is all points *above* the parabola $y = x^2$. It is sketched on the right.



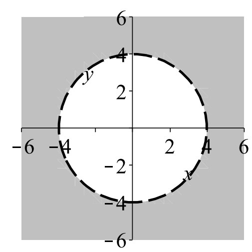
37. This is all points *between* the vertical lines $x = -7$ and $x = 7$ and *outside* of the horizontal lines $y = -5$ and $y = -3$. See the picture.



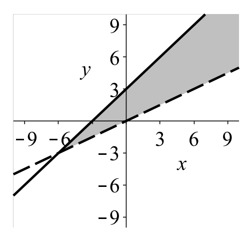
38. This is all points *on and between* the vertical lines $x = -5$ and $x = 5$ and *outside* of the horizontal lines $y = -2$ and $y = 2$. See the picture.



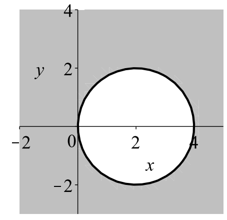
39. This is all points *outside* of the circle centered at the origin having radius 4. See the picture.



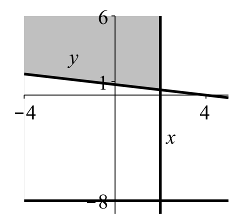
40. This is all points that are *above* the line $y = x/2$ and *on or below* the line $y = x + 3$. See the picture.



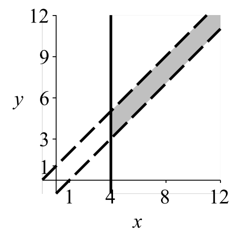
41. This is all points *on and outside* of the circle centered at (2,0) having radius 2. See the picture.



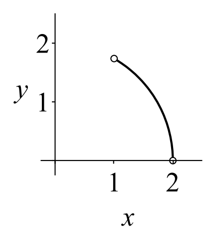
42. This is all points that are *on or above* the line $x + 5y = 4$, *on or left* of the vertical line $x = 2$, and *on or above* the horizontal line $y = -8$. See the picture.



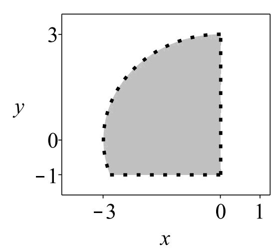
43. This is all points that are *between* the lines $x - y = 1$ and $x - y = -1$ and *on or to the right* of the vertical line $x = 4$. See the picture.



44. This are the points that are on the circle centered at the origin of radius 2 and in the first quadrant, to the right of the vertical line $x = 1$. See the picture.



45. This are points that are in the *left half-plane*, *inside* of the circle centered at the origin of radius 3, that also lie *above* the horizontal line $y = -1$. See the picture.



Further Theory and Practice

46. The points (2,3) and (8,10) are $d = \sqrt{36 + 49} = \sqrt{85}$ units apart. Therefore, we look for a point (x, y) on the intersection of the circles

$$(x - 2)^2 + (y - 3)^2 = r^2 \quad \text{and} \quad (x - 8)^2 + (y - 10)^2 = r^2$$

for any $r \geq \sqrt{85}/2$. This implies that $-4x + 4 - 6y + 9 = -16x + 64 - 20y + 100$ or $12x + 14y = 151$. Actually, any point on this line will serve the stated purpose (draw a picture). For example, $(5, 13/2)$, which is the point exactly half-way between (2,3) to (8,10).

47. The points equidistant from (2,3) and (8,2) are on the intersection of circles

$$(x - 2)^2 + (y - 3)^2 = r^2 \quad \text{and} \quad (x - 8)^2 + (y - 2)^2 = r^2 \,.$$

This implies that $-4x+4-6y+9=-16x+64-4y+4$ or $12x-2y=55$.

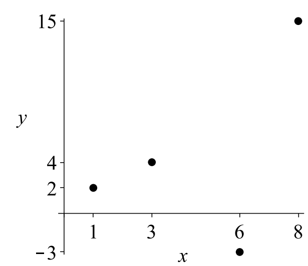
Similarly, the points equidistant from (2,3) and (7,9) are on the intersection of circles

$$(x-2)^2+(y-3)^2=r^2 \quad \text{and} \quad (x-7)^2+(y-9)^2=r^2\,.$$

This implies that $-4x+4-6y+9=-14x+49-18y+81$ or $10x+12y=117$.

Any point equidistant from the three points must lie on the intersection of these two lines. Solve simultaneously to find that the point is $(447/82, 427/82)$. Verify that the distance from this point to all three points given above is $5\sqrt{4514}/82$.

48. The picture on the right shows that the point equidistant from (1,2), (3,4), and $(6,-3)$ lies to the left of the vertical line $x=6$ and the point that is equidistant from (3,4), $(6,-3)$, and (8,15) lies to the right of that line.



49. There is just one such point, the origin (0,0). It is one unit from all three.

50. This is the equation of a hyperbola because x^2 and y^2 appear on the left side with opposite signs.

51. This wants to be the equation of a circle. Complete the square in x and y: $5(x^2-\frac{2}{5}x+\frac{1}{25})+5(y^2+\frac{3}{5}y+\frac{9}{100})=6+\frac{13}{20}$ or $(x-1/5)^2+(y+3/10)^2=133/100$, to see that it is.

52. Write the equation in the form $x^2+6x+y^2=-7$ and it might define a circle. Complete the square in x: $(x^2+6x+9)+y^2=-7+9$ or $(x+3)^2+y^2=2$, and indeed it is the equation of a circle.

53. This can be written in the form $2y^2-2x^2+x-5y=7$, which is the equation of a hyperbola.

54. Complete the square in x: $(x^2-16x+64)+y^2=64-k$ or $(x-8)^2+y^2=64-k$. If $k=64$ the equation is $(x-8)^2+y^2=0$ and its graph is the point (8,0).

55. Divide the equation by 2 and complete the square in x: $(x^2+9x+81/4)+y^2=-k/2+81/4$. Let $k=81/2$ and the equation is $(x+9/2)^2+y^2=0$ whose graph is the point $(-9/2,0)$.

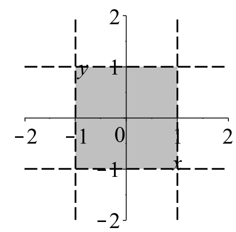
56. Complete the square in x and y: $(x^2-6x+9)+(y^2+2y+1)=1-k+9+1$. If $k=11$ then $(x-3)^2+(y+1)^2=0$. The graph is the point $(3,-1)$.

57. Write the equation in the form $x^2 + x + y^2 - y = k - 1$ and complete the square in x and y: $(x^2+x+1/4)+(y^2-y+1/4) = k-1+1/2$. If $k = 1/2$ then $(x + 1/2)^2 + (y - 1/2)^2 = 0$. The graph is the point $(-1/2, 1/2)$.

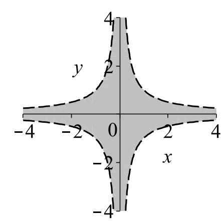
58. Complete the square in x and y: $(x^2 - 8x + 16) + (y^2 + 2y + 1) = k + 17$ or $(x - 4)^2 + (y + 1)^2 = k + 17$. The graph is empty for all $k < -17$.

59. Put the equation of the hyperbola in standard form (complete the square in x): $(x^2 + 6x + 9) - 4y^2 = 7 + 9$ or $\frac{(x+3)^2}{4^2} - \frac{y^2}{2^2} = 1$. The center of the hyperbola is the point $(-3, 0)$ and its vertices are at the points $(-3\pm4, 0)$. The values $a = -7$, $b = 1$ produce the widest open vertical strip containing no point of the hyperbola.

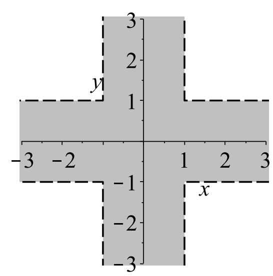
60. This is a square, centered at the origin. The boundary lines are not included in the region. See the picture.



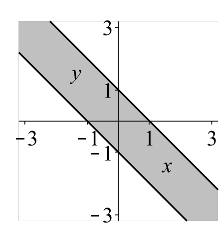
61. The four dashed curves in the picture are the points (x, y) where $|x| \cdot |y| = 1$. The points where $|x|\cdot|y| < 1$ are *inside* of these curves. The boundary lines are not included.



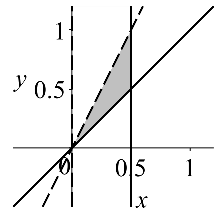
62. This is the union of two infinite strips, one horzontal and one vertical. The boundary lines are not included in the region. See the picture.



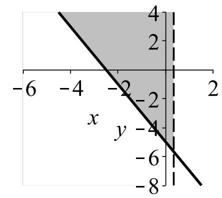
63. This is the region consisting of the points that lie *on and between* the two lines in the picture. The top line is $x + y = 1$ and the bottom line is $x + y = -1$.



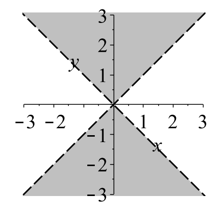
64. This region consists of the points in the first quadrant that lie *below* the line $y = 2x$ and on or above the line $y = x$ and on or to the left of the vertical line $x = 1/2$. See the picture.



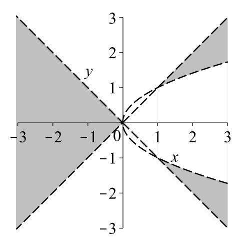
65. This region consists of the points that lie to the *left* of the vertical line $x = 1/3$ and *on or above* the line $y = -2x - 5$. See the picture.



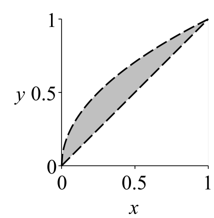
66. These are the points in the shaded region that are between the crossing lines $y = x$ and $y = -x$. See the picture.



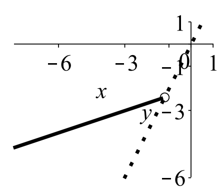
67. These are the points that lie *outside* of the "cup" of the parabola $x = y^2$ and, at the same time, lie between the crossing lines $y = x$ and $y = -x$ as shown in the picture.



68. These points lie in the first quadrant *above* the line $y = x$ and *below* the curve $y = \sqrt{x}$. See the picture.



69. These are the points that lie *on* the line $x - 3y = 6$ and are *above* the line $y = 2x$. See the picture.



70. Each circle has an equation of the form $(x-h)^2+(y-k)^2 = 65$. Substitute the coordinates of the two points to get two equations in the variables h and k: $h^2 + (-6 - k)^2 = 65$ and $(3 - h)^2 + (-5 - k)^2 = 65$. Equate the

left hand sides to obtain $12k + 36 = 9 - 6h + 10k + 25$ or $2k + 6h = -2$. This implies that $k = -1 - 3h$. Substituting $-1 - 3h$ for k into the first circle equation yields $h^2 + (-6 + 1 + 3h)^2 = 65$ which expands to $10h^2 - 30h + 25 = 65$ or $h^2 - 3h - 4 = 0$ so $(h - 4)(h + 1) = 0$ and $h = 4$ or $h = -1$. If $h = 4$, then $k = -13$ and if $h = -1$, then $k = 2$. The two centers are $(4, -13)$ and $(-1, 2)$.

71. The intersection consists of points (x, y) such that $y = 2x$ and $(x - h)^2 + (y - k)^2 = r^2$. Substitute the first equation into the second to see that x must be a solution to $(x - h)^2 + (2x - k)^2 = r^2$, a quadratic in x. There will be either no solutions, exactly one solution, or exactly two solutions.

72. Let the equation of the circle be $(x - h)^2 + (y - k)^2 = r^2$. Substitute the coordinates of the first two points into this equation and set the left sides equal to obtain $6h + 9 - 8k + 16 = -2h + 1 - 12k + 36$, which simplifies to $2h + k = 3$. Now substitute the coordinates of the second and third points into the equation, equate left sides, and simplify to $16h - 12k = 44$ or $4h - 3k = 11$. Subtract twice $2h + k = 3$ from $4h - 3k = 11$ to see that $k = -1$ (and $h = 2$). To obtain the radius of the circle let $x = 9$ and $y = 0$ in the equation $(x - 2)^2 + (y + 1)^2 = r^2$ to get $49 + 1 = r^2$ so $r = \sqrt{50}$.

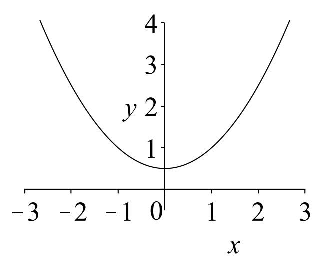
73. Let the equation of the circle be $(x - h)^2 + (y - k)^2 = r^2$. Substitute the coordinates of the first two points into this equation and set the left sides equal to obtain $-4h + 4 - 8k + 16 = -8h + 16$, which simplifies to $4h - 8k = -4$ or $h - 2k = -1$. Now substitute the coordinates of the second and third points into the equation, equate left sides, and simplify to $18h + 6k = -18$ or $3h + k = -3$. Add twice $3h + k = -3$ to $h - 2k = -1$ to see that $h = -1$ (and $k = 0$). To obtain the circle's radius, let $x = 4$ and $y = 0$ in the equation $(x + 1)^2 + y^2 = r^2$ to get $25 = r^2$ so $r = 5$.

74. Interchange the roles of x and y in Theorem 1 to see that the horizontal line $y = -\beta/(2\alpha)$ is the axis of symmetry of the parabola. The vertex is at the point (x_0, y_0) where $y_0 = -\beta/(2\alpha)$, and $x_0 = \alpha(-\beta/(2\alpha))^2 + \beta(-\beta/(2\alpha)) + \gamma = (4\alpha\gamma - \beta^2)/(4\alpha)$.

75. Let $P = (x, y)$. Then

$$y = \sqrt{(x - 0)^2 + (y - 1)^2},$$

implying that $y \geq 0$ and $y^2 = x^2 + (y - 1)^2$. That is, $y \geq 0$ and $0 = x^2 + 1 - 2y$. Equivalently, $y = (x^2 + 1)/2$. This is a parabola. See the picture.



76. Assume that $A > 0$. The x-coordinates of the points of intersection are the solutions to the quadratic equation $Ax^2 + Bx + C - y_0 = 0$. That is,

$$x_0 = \frac{-B - \sqrt{B^2 - 4A(C - y_0)}}{2A} \quad \text{and} \quad x_1 = \frac{-B + \sqrt{B^2 - 4A(C - y_0)}}{2A}.$$

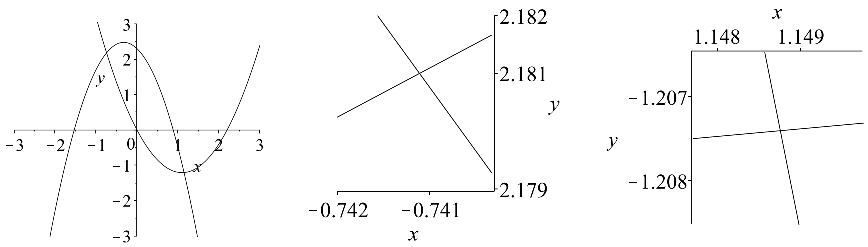
Since $x_0 < s < x_1$, $|x_0 - s| = s - x_0 = \frac{\sqrt{B^2-4A(C-y_0)}}{2A}$ and $|x_1 - s| = x_1 - s = \frac{\sqrt{B^2-4A(C-y_0)}}{2A}$.

If $A < 0$, then x_0 and x_1 switch places, and

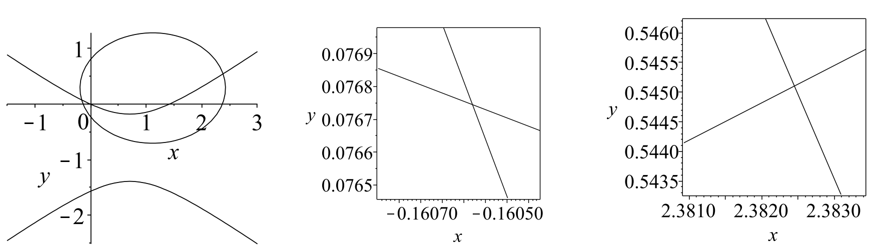
$$|x_0 - s| = |x_1 - s| = \frac{\sqrt{B^2 - 4A(C - y_0)}}{-2A} = \frac{\sqrt{B^2 - 4A(C - y_0)}}{2|A|}.$$

CALCULATOR/COMPUTER EXERCISES

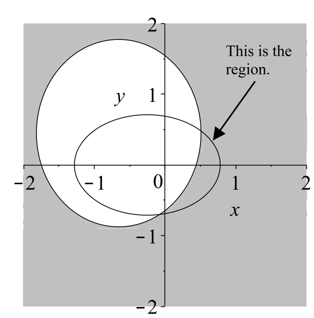
77. These are both parabolas. The picture on the left displays them from a distance. The zoomed versions show that they intersect at the points $(-0.741, 2.181)$ and $(1.149, -1.207)$.



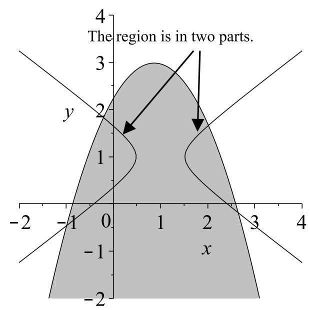
78. This is an ellipse and a hyperbola. The picture on the left displays them from a distance. The zoomed versions show that they intersect at the points $(-0.161, 0.077)$ and $(2.383, 0.545)$.



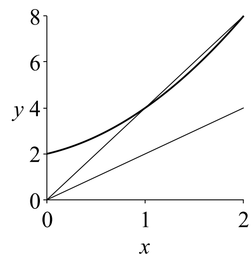
79. The boundary curves are ellipses. The region is all points on and inside the smaller one and on and outside the larger. See the picture.

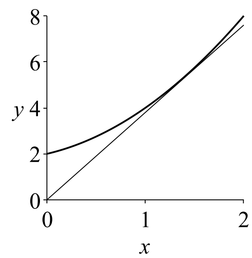


80. The region consists of the points *on and below* the parabola and *inside* the two branches of the hyperbola. It splits naturally up into two parts. See the picture.



81. The parabola and the two lines are sketched below on the left. On the right is the same parabola and the line $y = 3.8x$.





1.3 Lines and Their Slopes

Problems for Practice

1. The line through (0,3) also passes through (4,4). Its slope is $\frac{4-3}{4-0} = \frac{1}{4}$.

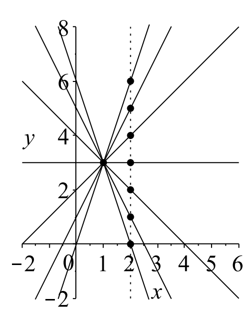
   One line through $(-1, -4)$ also passes through $(3, -4)$. Its slope is 0.

   Another line through $(-1, -4)$ goes through $(-3, 0)$, slope: $\frac{0-(-4)}{-3-(-1)} = -2$.

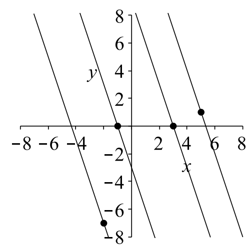
   The line through $(-5, 4)$ also passes through (4,0), slope: $\frac{0-4}{4-(-5)} = -\frac{4}{9}$.

   The line through (3,0) also passes through $(3, -4)$. It is vertical and does not have a slope.

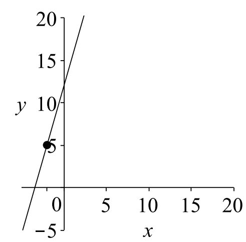
2. The lines are drawn on the right. The vertical dotted line is at the point $x = 2$.



3. The lines are drawn on the right. All of them have slope -3.



4. The line is drawn on the right. Its slope is $7/2$.



5. $y = 5(x+3)+7$

6. $y = -2(x-4.1)+8.2$

7. $y = -(x+\sqrt{2}\,)$

8. $y = \pi$

9. The slope is $(-4-7)/(6-2) = -11/4$.
   Using the first point, $y = (-11/4)(x-2)+7$.

10. The slope is $(-4-1)/(-4-12) = 5/16$.
    Using the first point, $y = (5/16)(x-12)+1$.

11. The slope is $(0-(-4))/(9-(-7)) = 1/4$.
    Using the second point, $y = (1/4)(x-9)$.

12. The slope is $(1-(-5))/(-5-1) = -1$.
    Using the first point, $y = -(x-1)-5$.

13. $y = -4x+9$

14. $y = \pi x + \pi^2$

15. $y = \sqrt{2}\,x - \sqrt{3}$

16. $y = 3$

17. Use the point $(-4,0)$ and the slope 3: $y = 3(x+4)+0$ or $y = 3x+12$.

18. Use the point $(5,0)$ and the slope -5: $y = -5(x-5)+0$ or $y = -5x+25$.

19. Using the intercept form: $\frac{x}{-2} + \frac{y}{6} = 1$. Therefore, $y = 3x+6$.

20. Using the intercept form: $\frac{x}{-5} + \frac{y}{-12} = 1$. Therefore, $y = (-12/5)x - 12$.

21. The slope is $(10-7)/(3-2)=3$. Using the point (2,7), $y=3(x-2)+7$ or $y=3x+1$.

22. The slope is $(7-1)/(2-1/2)=4$. Using the point (2,7), $y=4(x-2)+7$ or $y=4x-1$.

23. The slope is $(-9-3)/(2-(-1))=-4$. Using the point $(-1,3)$, $y=-4(x+1)+3$ or $y=-4x-1$.

24. The slope is $(0-(-\pi))/(0-2\pi)=-1/2$. Using the point (0,0), $y=(-1/2)x$ or $y=-x/2$.

25. $\dfrac{x}{-2}+\dfrac{y}{6}=1$

26. $\dfrac{x}{5}+\dfrac{y}{1/3}=1$

27. $\dfrac{x}{-1}+\dfrac{y}{3}=1$

28. $\dfrac{x}{5/2}+\dfrac{y}{5}=1$

29. The line has slope $-1/2$ so $y=(-1/2)(x-1)-2$ or $y=-x/2-3/2$.

30. The line has slope $-2/3$ so $y=(-2/3)(x-3)+4$ or $y=-2x/3+6$.

31. The line has slope 2 so $y=2(x-2)+1$ or $y=2x-3$.

32. The line has slope $-1/2$ so $y=(-1/2)(x-4)$ or $y=-x/2+2$.

33. The given line has slope $-1/2$ so the line perpendicular to it has slope 2. $y=2(x-1)-2$ or $y=2x-4$.

34. The given line has slope 3 so the line perpendicular to it has slope $-1/3$. $y=(-1/3)(x-0)-5$ or $y=-x/3-5$.

35. The given line has slope $-2/3$ so the line perpendicular to it has slope $3/2$. $y=(3/2)(x-3)+4$ or $y=3x/2-1/2$.

36. The given line has slope $-1/6$ so the line perpendicular to it has slope 6. $y=6(x+3)+0$ or $y=6x+18$.

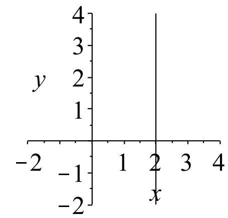
37. The x-intercept is $2/3$ and the y-intercept is $1/2$. The slope of the line is $-3/4$.

38. The x-intercept is 8 and the y-intercept is 2. The slope of the line is $-1/4$.

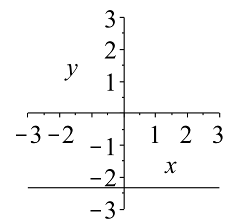
39. The x-intercept is 3 and the y-intercept is -1. The slope of the line is $1/3$.

40. The x-intercept is 2 and the y-intercept is -5. The slope of the line is $5/2$.

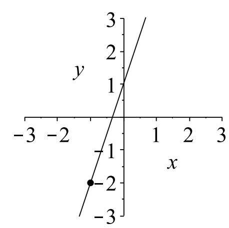
41. The line with equation $x/2 = 1$ is vertical, with x-intercept $x = 2$. See the picture.



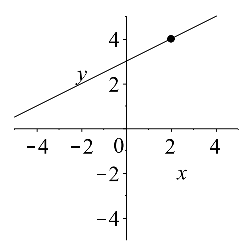
42. The line with equation $3y = -7$ is horizontal, with y-intercept $y = -7/3$. See the picture.



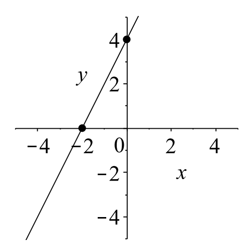
43. This line has slope 3 and it passes through the point $(-1, -2)$. See the picture.



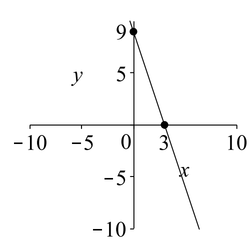
44. This line has slope $1/2$ and it passes through the point $(2, 4)$. See the picture.



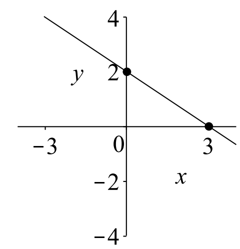
45. This line has intercepts $(-2, 0)$ and (0,4). Its slope is 2. See the picture.



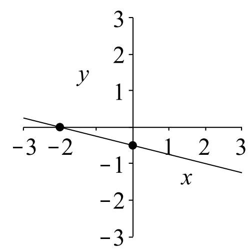
46. This line has intercepts $(3, 0)$ and (0,9). Its slope is -3. See the picture.



47. This line has intercepts $(3, 0)$ and (0,2). Its slope is $-2/3$. See the picture.



48. This line has intercepts $(-2, 0)$ and $(0, -1/2)$. Its slope is $-1/4$. See the picture.



49. The line in question has slope $(7-2)/(4-1) = 5/3$. It is not parallel to the line through $(-4, -3)$ and $(-7, 6)$ because that line has slope $9/(-3) = -3$. Nor is it perpendicular to the line through (7,9) and $(5, -7)$ because that line has slope $-16/(-2) = 8$.

50. The line in question has slope $(23 - 8)/(2 - (-3)) = 3$. It is not parallel to the line through $(-4, 6)$ and $(3, 28)$ because that line has slope $22/7$. It is perpendicular to the line through $(-2, 4)$ and $(7, 1)$ because that line has slope $-1/3$, the negative reciprocal of 3.

51. The line in the figure has slope $-4/12 = -1/3$. Therefore, the line through $(7, -8)$ that is perpendicular to it has slope 3 and it has the following equation: $y = 3(x - 7) - 8$ or $y = 3x - 29$.

52. The two blue lines are parallel, each having slope $1/4$. The black line that appears to be perpendicular to the blue ones is. Its slope is -4. The other black line passes through $(-2, -3)$ and $(1, 2)$ and its slope is $5/3$.

Further Theory and Practice

53. The line in question has the equation $y = 2(x-5)-7$. It will pass through (a, b) when, for example, $a = 6$ and $b = -5$.

54. The slope of the line in question is $(9 - 6)/(2 - 3) = -3$ so its equation is $y = -3(x - 4) + 2$. The point (a, b) will be on this line when, for example, $a = 5$ and $b = -1$.

55. Since it is perpendicular, the slope of the line in question is the negative reciprocal of $(9-7)/(4-(-2)) = 1/3$ so its equation is $y = -3(x+2)+7$. The point (a, b) will be on this line when, for example, $a = -1$ and $b = 4$.

56. The coordinates of the point (a,b) must satisfy the following two equations:

$$(a-4)^2+(b-6)^2=1 \quad \text{and} \quad \frac{b-6}{a-4}=-\frac{12}{2}\,.$$

Thus, $b-6=-6(a-4)$, $(a-4)^2+36(a-4)^2=1$, and $(a-4)^2=1/37$. Consequently, $a=4\pm\sqrt{37}/37$ and $b=6\mp 6\sqrt{37}/37$. There are two points: $(4+\sqrt{37}/37, 6-6\sqrt{37}/37)$ and $(4-\sqrt{37}/37, 6+6\sqrt{37}/37)$.

57. Let (a,b) lie in the first quadrant, on the unit circle ($a^2+b^2=1$), and on the line $y=2x$. Then $b=2a$ and $a^2+4a^2=1$, so $a=\sqrt{5}/5$ and $b=2\sqrt{5}/5$. Observe that the point $(c,d)=(5a,5b)=(\sqrt{5}\,,2\sqrt{5}\,)$ is 5 units from the origin and also lies on the line $y=2x$.

58. The point (a,b) is on the line $y=\left(\frac{7-(-4)}{-2-9}\right)\cdot(x+2)+7$ or $y=-x+5$ and on the line $y=8(x-2)+1$ or $y=8x-15$. Consequently, $-a+5=8a-15$, so $a=20/9$ and $b=5-a=25/9$.

59. Subtract twice the second equation from the first to obtain $-5y=-10$. Therefore, $y=2$ and $x=7-3y=1$.

60. Substitute $x=y+2$ into the first equation: $3(y+2)+5y=14$ or $8y=8$. Therefore, $y=1$ and $x=y+2=3$.

61. Add 5 times equation 1 to 2 times equation 2 to obtain $29y=190-16$ or $y=6$. Substitute this into equation 2: $5x=-8-12$, so $x=-4$.

62. Subtract the second equation from the first to obtain $0=x-1/2$. Therefore, $x=1/2$ and $y=2x+1=2$.

63. The point (a,b) that is nearest to (2,8) must satisfy two equations:

$$3a-8b=4 \quad \text{and} \quad \frac{b-8}{a-2}=-\frac{8}{3}\,.$$

The second equation reflects the fact that the line from (a,b) to $(2,8)$ must be perpendicular to the line $3x-8y=4$. Therefore, $3a-8b=4$ and $3b-24=-8a+16$ or $8a+3b=40$. Add 3 times the first equation to 8 times the second to obtain $73a=332$ or $a=332/73$. Substitute this into either of the two equations to see that $b=88/73$.

64. Substitute $x=7y-15$ into the circle equation: $(7y-18)^2+(y+1)^2=25$ or $50y^2-250y+325=25$. That is, $y^2-5y+6=0$. Therefore, $(y-2)(y-3)=0$ and $y=2$ or $y=3$. If $y=2$, then $x=-1$ and if $y=3$, then $x=6$. The intersection points are $(-1,2)$ and $(6,3)$.

65. If $A=0$, then the first line is horizontal and the second one is vertical. If $B=0$, then this is reversed. In either case, the lines are perpendicular. In the case that both A and B are not zero, then the slope of the first line is $-A/B$ and the slope of the second line is B/A. Once more, the lines are perpendicular.

66. The slope of the ramps should be no more than 1 inch per foot. There are $8 \cdot 12 = 96$ inches in 8 feet. Therefore, the wheelchair ramp must be at least 96 feet long. This can be accomplished by building it in three 30 foot straight ramps that go from the street to the building then back to the street and to the building again. This will attain a height of 90 inches. The final 6 inches can be added in a 6 foot extension from the end of the final ramp to the front door.

67. The line $y = y_0$ meets the first line at $x_1 = (y_0 - 1)/2$ and it meets the second line at $x_2 = y_0 - 2$. The distance from one intersection point to the other is $d = |x_2 - x_1| = |y_0/2 - 3/2|$. The values of y_0 for which $d = 10^6$ are the solutions to the equation $|y_0 - 3| = 2 \cdot 10^6$. That is, $y_0 = 3 + 2 \cdot 10^6$ and $y_0 = 3 - 2 \cdot 10^6$.

68. The slope of the regression line through $P_0 = (x_0, y_0)$ is

$$m = \frac{(x_1 - x_0)(y_1 - y_0) + (x_2 - x_0)(y_2 - y_0)}{(x_1 - x_0)^2 + (x_2 - x_0)^2} = \frac{(-h)(u - y_0) + h(v - y_0)}{h^2 + h^2} = \frac{v - u}{2h}\,.$$

This is the average of the slopes of the seqments P_1P_0 and P_0P_2.

69. The budget line has a Cartesian equation of the form $p_X \cdot x + p_Y \cdot y = C$. Its intercepts are $(C/p_X, 0)$ and $(0, C/p_Y)$ and its slope is $-p_X/p_Y$. The new budget line is parallel to the old one. If $C' > C$, then the new line lies above the old one. Otherwise, it lies below it.

70. The line through (a, a^2) and (b, b^2) has slope $(b^2 - a^2)/(b - a) = b + a$. Its equation is $y = (b + a)(x - a) + a^2$. The midpoint of $\overline{PQ}$ is the point $((a+b)/2, (a^2+b^2)/2)$. It is above the parabola if and only if $(a^2+b^2)/2 > ((a+b)/2)^2$. That is, $a^2+b^2 > (a+b)^2/2$ or $a^2+b^2 > a^2/2+ab+b^2/2$. But this inequality is equivalent to $a^2/2 - ab + b^2/2 > 0$ or $a^2 - 2ab + b^2 > 0$. That is, $(a - b)^2 > 0$, which it true provided that $a \neq b$.

71. Lines perpendicular to ℓ have an equation of the form $-Bx + Ay = C'$. Therefore, the line perpendicular to ℓ that passes through the point (x_0, y_0) has $C' = -Bx_0 + Ay_0$, and intersects ℓ at the point (x_1, y_1) where $x = x_1$ and $y = y_1$ are the solutions to the following two equations.

$$\begin{array}{rcrcl} Ax & + & By & = & C \\ -Bx & + & Ay & = & C' \end{array}$$

Multiply the first equation by A, the second by $-B$, and add to see that

$$\begin{aligned}(A^2 + B^2)x_1 = AC - BC' &= AC - B(-Bx_0 + Ay_0) \\ &= AC + B^2x_0 - ABy_0\,.\end{aligned}$$

Similarly, $(A^2+B^2)y_1 = AC' + BC = BC + A^2y_0 - ABx_0$. Therefore, the distance from ℓ to (x_0, y_0) is the square root of the following expression.

$$(x_1 - x_0)^2 + (y_1 - y_0)^2 = \left(\frac{AC + B^2x_0 - ABy_0 - x_0(A^2+B^2)}{A^2+B^2}\right)^2 + \left(\frac{BC + A^2y_0 - ABx_0 - y_0(A^2+B^2)}{A^2+B^2}\right)^2 = \frac{(Ax_0 + By_0 - C)^2}{A^2+B^2}$$

The details of the simplification of line 1 to line 2 are left to the reader.

72. If $P = (a, b)$ and $Q = (c, d)$ are lattice points, and the midpoint of the line segment $\overline{PQ}$: $((a+b)/2, (c+d)/2)$, is also a lattice point, then $a+b$ and $c+d$ must be evenly divisible by 2. That is, $a+b$ and $c+d$ must be even. Consequently, either a and b are both odd or they are both even, and the same thing is true for c and d.

    Given two distinct lattice points $P = (a, b)$ and $Q = (c, d)$ there are at most 8 different ways to assign parities ("odd" or "even") to the coordinates a, b, c and d. If there are 5 distinct lattice points, then there are 10 distinct line segments (verify) and since there are no more than 8 ways to assign parities, at least one segment will have endpoints of the same parity and its midpoint will be a lattice point.

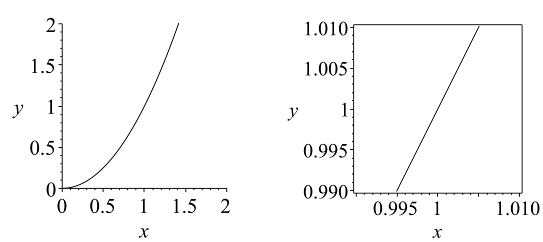
73. Assuming that $s \neq t$, the line ℓ' has slope $(s-t)/(t-s) = -1$. Therefore, ℓ' is perpendicular to the line $y = x$ which has slope 1. The intersection of $y = x$ and ℓ' is the point (a, a) such that $x = a$ and $y = a$ satisfies the equation $y = -(x-s)+t$. Thus $a = -(a-s)+t$ and $2a = s+t$ so $a = (s+t)/2$. The distance squared from (a, a) to (s, t) is

$$(a-s)^2 + (a-t)^2 = ((t-s)/2)^2 + ((s-t)/2)^2 = (s-t)^2/2\,.$$

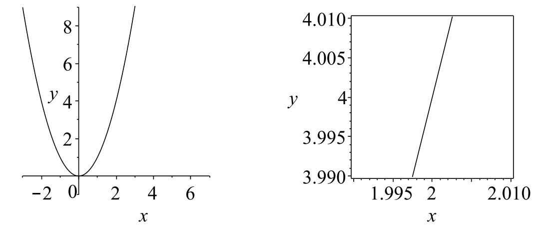
    This is the same as the distance squared from (a, a) to (t, s) (verify).

CALCULATOR/COMPUTER EXERCISES

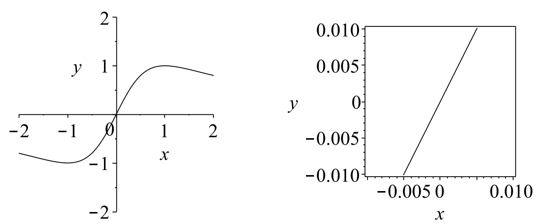
74. The plot on the left is centered at the point (1,1). The right plot is centered at the same point, but zoomed in. The slope appears to be $m = 2$.



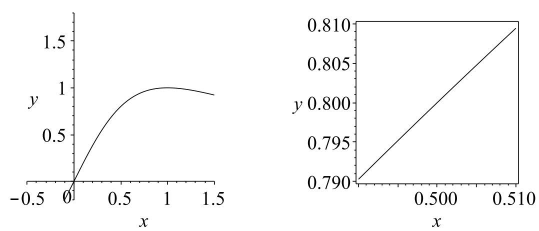
75. The plot on the left is centered at the point (2,4). The right plot is centered at the same point, but zoomed in. The slope appears to be $m = 4$.



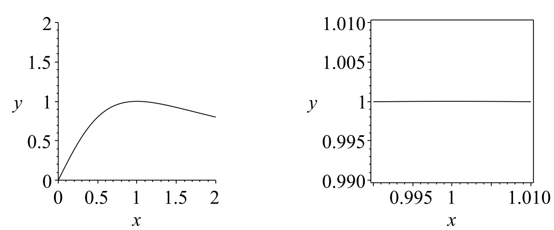
76. The plot on the left is centered at the point (0,0). The right plot is centered at the same point, but zoomed in. The slope appears to be $m = 2$.



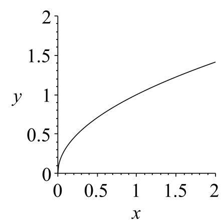
77. The plot on the left is centered at the point (1/2,4/5). The right plot is centered at the same point, but zoomed in. The slope appears to be $m = 1$.

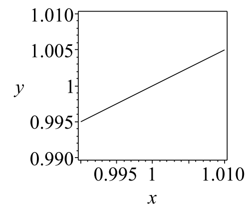


78. The plot on the left is centered at the point (1,1). The right plot is centered at the same point, but zoomed in. The slope appears to be $m = 0$.



79. The plot on the left is centered at the point (1,1). The right plot is centered at the same point, but zoomed in. The slope appears to be $m = 1/2$.

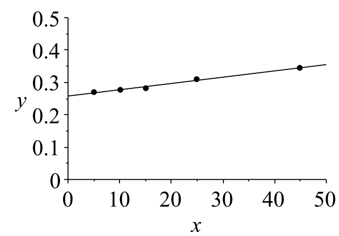




80. The points and the regression line are sketched on the right. The regression line equation is

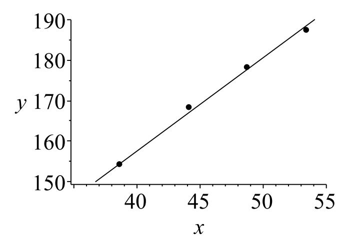
$$y = 0.001942x + 0.257865\,.$$

Substituting $x = 100$ yields a predicted value of $y = 0.452$ grams per mile.



81. The points and the regression line are sketched on the right. The regression line equation is

$$y = 2.309451x + 65.155206\,.$$

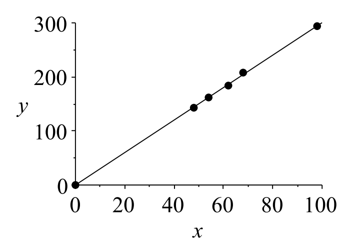


82. The following table contains the x and y data.

(width,height)	(7,42)	(8,23)	(11,13)	(18,9)	(26,8)
x (perimeter)	98	62	48	54	68
y (area)	294	184	143	162	208

The points (x, y) and the regression line through (0,0) are displayed on the right. The regression line equation is

$$y = 3.004293x\,.$$



The regression line approximation of y when $x = 8$ is 24.034. Since the actual y value is 4, this is a relative error of approximately $100 \times \left|\frac{24-4}{4}\right| = 500\%$.

Nothing went wrong. A handful of specially chosen rectangles yielding data points that lie close to a line should not be used to formulate a general rule relating y to x.

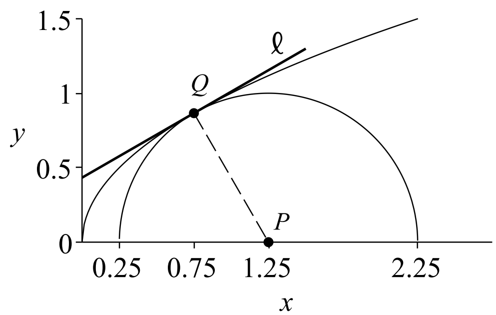
83. The equation of the top half of the circle centered at $(5/4, 0)$ of radius 1 is $y = \sqrt{1 - (x - 5/4)^2}$. It intersects the curve $y = \sqrt{x}$ when $x = 1 - (x - 5/4)^2$. That is, $x = 1 - x^2 + (5/2)x - 25/16$, or $x^2 - (3/2)x + 9/16 = 0$. This

simplifies to $(x-3/4)^2 = 0$ and the intersection point is $Q = (3/4, \sqrt{3}/2)$. The equation of the line joining P to Q is $y = \frac{\sqrt{3}/2}{3/4-5/4} \cdot (x - 5/4)$ or

$$y = -\sqrt{3}\,(x - 5/4)\,.$$

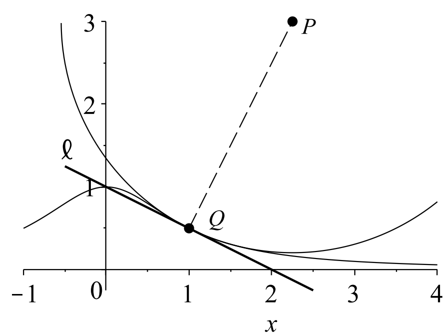
It is sketched on the right along with the half-circle and the line ℓ through Q that is perpendicular to line $\overline{PQ}$. The equation for the line ℓ is

$$y = (\sqrt{3}/3)(x - 3/4) + \sqrt{3}/2\,.$$



84. The equation of the bottom half of the circle centered at $(9/4, 3)$ of radius $5\sqrt{5}/4$ is $y = 3 - \sqrt{125/16 - (x - 9/4)^2}$. According to *Maple* it intersects the curve $y = 1/(1 + x^2)$ when $x = 1$ and when $x = (3^{2/3} + 1)/2$. The intersection that yields a tangent line is $x = 1$. The curves intersect at the point $Q = (1, 1/2)$. The equation of the line joining P to Q is $y = 2(x-1)+1/2$ or $y = 2x-3/2$. It is sketched on the right along with a portion of the half-circle and the line ℓ through Q that is perpendicular to line $\overline{PQ}$. The equation for the line ℓ is

$$y = -x/2 + 1\,.$$

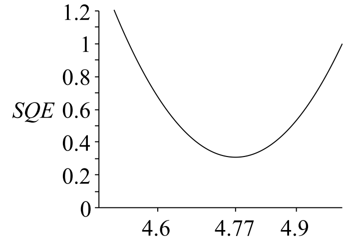


85. The line through P_0 with slope m is $y = m \cdot (x - 4) + 16$. The sum of the squares of the errors associated with this line is

$$\begin{aligned} d_1^2 + d_2^2 &= (m(1-4) + 16 - 2)^2 + (m(2-4) + 16 - 6)^2 \\ &= 13m^2 - 124m + 296\,. \end{aligned}$$

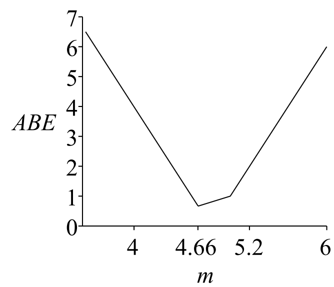
The graph of $SQE = d_1^2 + d_2^2$ versus m is displayed on the right. Based upon this picture we estimate the slope of the regression line to be $m_0 = 4.77$. The equation of the regression line $\mathcal{L}$ would then be

$$y = 4.77(x - 4) + 16\,.$$

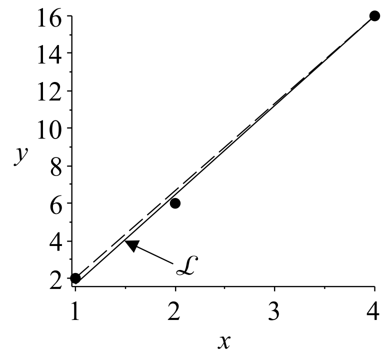


The graph of $ABE = d_1 + d_2$ versus m is displayed on the right. Based upon this picture we estimate the slope of the line minimizing ABE to be $m_1 = 4.66$. The equation of the line is

$$y = 4.66(x - 4) + 16\,.$$



The picture on the right displays the three data points and the two lines. The one minimizing the sum of the squared errors is the solid line $\mathcal{L}$ and the one minimizing the sum of the absolute errors is the dashed line.

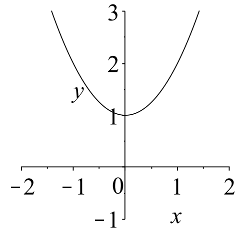


1.4 Functions and Their Graphs

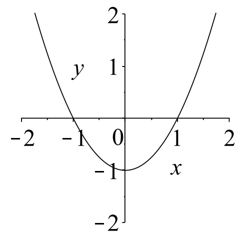
Problems for Practice

1. Division by 0 is not allowed, so the domain is all $x \neq -1$.
2. $x^2 + 2 > 0$ for all x, so the domain is all of $\mathbb{R}$.
3. It must be the case that $x^2 - 2 \geq 0$, so $x^2 \geq 2$ or $|x| \geq \sqrt{2}$. The domain is $(-\infty, -\sqrt{2}\,] \cup [\sqrt{2}\,, \infty)$.
4. It must be the case that $2 - x^2 \geq 0$, so $x^2 \leq 2$ or $|x| \leq \sqrt{2}$. The domain is $[-\sqrt{2}\,, \sqrt{2}\,]$.
5. To avoid division by 0, $x^2 \neq 1$. The domain: $\{x \in \mathbb{R} : x \neq 1 \text{ and } x \neq -1\}$.
6. To be sure that the numerator is real, $x \geq 0$. In addition, since the denominator is $(x-2)(x+3)$, the domain must exclude $x = 2$. Therefore, it is $\{x \in \mathbb{R} : x \geq 0 \;\text{ and }\; x \neq 2\}$.
7. The quadratic $x^2 - 4x + 5$ has no real roots and it evaluates to a positive number for all x. Therefore, the domain is $\mathbb{R}$.
8. The domain will consist of all x such that $(x^2 - 4)(x - 1) > 0$. Therefore, if $x > 1$, then x^2 must be greater than 4 yielding the interval $(2, \infty)$. Moreover, if $x < 1$, then x^2 must be less than 4 yielding the interval $(-2, 1)$. The domain is $(-2, 1) \cup (2, \infty)$.

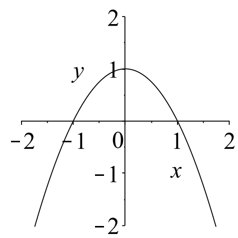
9. The graph appears on the right.



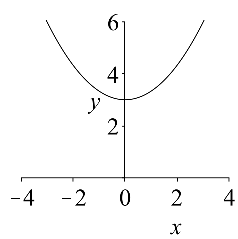
10. The graph appears on the right.



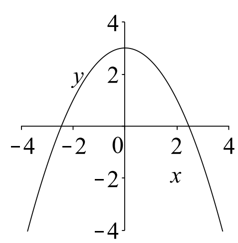
11. The graph appears on the right.



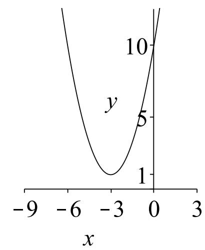
12. The graph appears on the right.



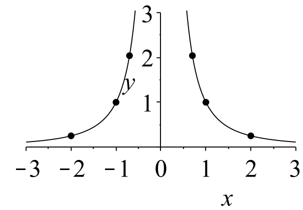
13. The graph appears on the right.



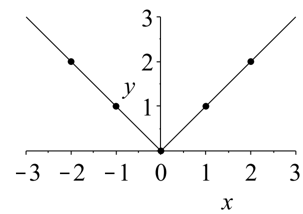
14. The graph appears on the right.



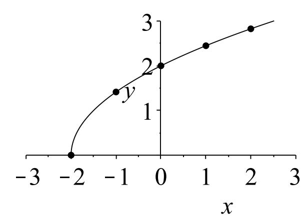
15. The graph appears on the right. The points are plotted for the x values $-2, -1, -0.7, 0.7, 1, 2$.



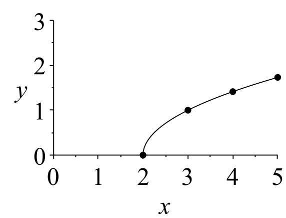
16. The graph appears on the right. The points are plotted for the x values $-2, -1, 0, 1, 2$.



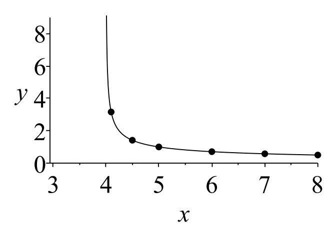
17. The graph appears on the right. The points are plotted for the x values $-2, -1, 0, 1, 2$.



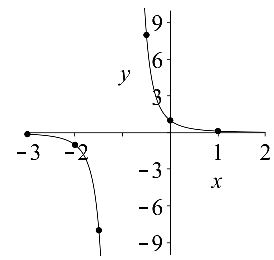
18. The graph appears on the right. The points are plotted for the x values $2, 3, 4, 5$.



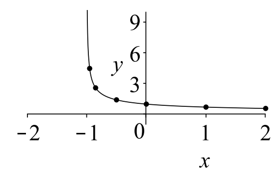
19. The graph appears on the right. The points are plotted for the x values $4.1, 4.5, 5, 6, 7, 8$.



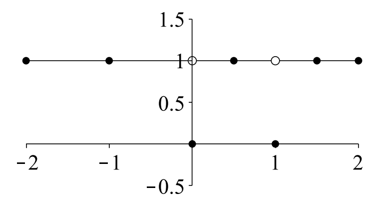
20. The graph appears on the right. The points are plotted for the x values $-3, -2, -1.5, -0.5, 0, 1$.



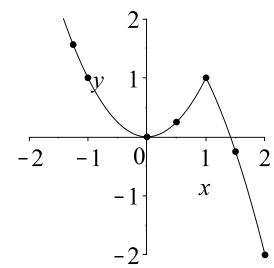
21. The graph appears on the right. The points are plotted for the x values $-0.95, -0.85, -0.5, 0, 1, 2$.



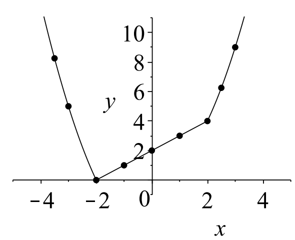
22. The graph appears on the right. The points are plotted for the x values $-2, -1, 0, 0.5, 1, 1.5, 2$.



23. The graph appears on the right. The points are plotted for the x values $-1.25, -1, 0, 0.5, 1, 1.5, 2$.



24. The graph appears on the right. The points are plotted for the x values $-3.5, -3, -2, -1, 0, 1, 2, 2.5, 3$.



Further Theory and Practice

25. Assuming $s(\alpha) = k\alpha$, since $s(360) = 2\pi r$, we conclude that $k \cdot 360 = 2\pi r$ so $k = \pi r/180$. Consequently, the formula for the arc length function s when the central angle is measured in degrees is $s(\alpha) = \pi r\alpha/180$.

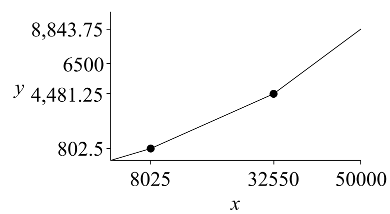
26. Since $f(x) = mx + b$, $\Delta y = f(x_0 + \Delta x) - f(x_0) = m \cdot (x_0 + \Delta x_0) + b - (mx_0 + b) = m\Delta x$. Than is, $\Delta y = m\Delta x$.

27. Assuming $A(r,\alpha) = kr\alpha$, since $A(r,360) = \pi r^2$, we may conclude that $k \cdot r \cdot 360 = \pi r^2$ so $k = \pi r/360$. Consequently, the formula for the area function A when the central angle is measured in degrees is $A(r,\alpha) = \pi r^2 \alpha/360$.

28. The height of the triangle is $\ell \cos(\alpha/2)$ and half of its base is $\ell \sin(\alpha/2)$. Therefore, its area is $A(\ell,\alpha) = \ell^2 \sin(\alpha/2)\cos(\alpha/2)$.

29. Observe that $x^2 + Ax + B = (x-r)(x-s) = x^2 - (r+s)x + rs$. Since this it true for all x, it follows that $A = -(r+s)$ and $B = rs$. That is, $A(r,s) = -(r+s)$ and $B(r,s) = rs$.

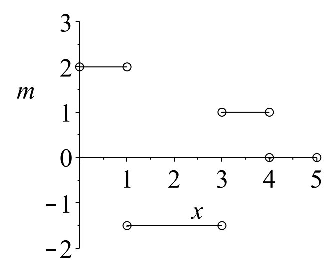
30. The graph appears on the right. The points are the break points for a change in the tax formula.



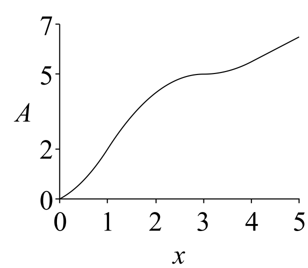
31. From $x = 0$ to $x = 1$, $f(x) = 2x + 1$. From $x = 1$ to $x = 3$, $f(x) = -(3/2)(x-3)$. From $x = 3$ to $x = 4$, $f(x) = x - 3$. From $x = 4$ to $x = 5$, $f(x) = 1$. This can be expressed in piecewise form as follows.

$$f(x) = \begin{cases} 2x+1 & \text{if } 0 \le x < 1 \\ -3(x-3)/2 & \text{if } 1 \le x < 3 \\ x-3 & \text{if } 3 \le x < 4 \\ 1 & \text{if } 4 \le x \le 5 \end{cases}$$

32. The domain of m is all x in the open interval $(0,5)$ except $x = 1, 3$, and 4. Its graph is skecthed on the right.



Summing the areas of trapazoids, the area function is $A(x) = \frac{1}{2}x(1+1+2x)$ from $x = 0$ to $x = 1$ where $A(1) = 2$. From $x = 1$ to $x = 3$, $A(x) = 2 + \frac{1}{2}(x-1)(3 - 3(x-3)/2)$ and $A(3) = 5$. From $x = 3$ to $x = 4$, $A(x) = 5 + \frac{1}{2}(x-3)(x-3)$ and $A(4) = 11/2$. From $x = 4$ to $x = 5$, $A(x) = 11/2 + (x-4)$.



33. Examination of the tax formulas produces the following piecewise formula

for the slope function.

$$m(x) = \begin{cases} 0.10 & \text{if} \quad 0 < x < 8025 \\ 0.15 & \text{if} \quad 8025 < x < 32550 \\ 0.25 & \text{if} \quad 32550 < x < 50000 \end{cases}$$

Since the areas accumulate by multiplying the height by the base, for $x = 0$ to $x = 8025$ the area is $A(x) = 0.10x$. The area at $x = 8025$ is 802.5. From $x = 8025$ to $x = 32550$ the area is $A(x) = 802.5 + 0.15(x - 8025)$ and the area at $x = 32550$ is $A(32550) = 4481.25$. From $x = 32550$ to $x = 50000$ the area function is $A(x) = 4481.25 + 0.25(x - 32550)$. In multicase form

$$A(x) = \begin{cases} 0.10x & \text{if} \quad 0 < x \leq 8025 \\ 802.5 + 0.15(x - 8025) & \text{if} \quad 8025 < x \leq 32550 \\ 4481.25 + 0.25(x - 32550 & \text{if} \quad 32550 < x \leq 50000 \end{cases}$$

Observe that $A(x) = T(x)$.

34. The first few terms are $2, 4, 6, 8, \cdots$. This can be generated by defining $f_1 = 2$ and $f_{n+1} = f_n + 2$.

35. The first few terms are $2, 4, 8, 16, \cdots$. This can be generated by defining $f_1 = 2$ and $f_{n+1} = 2 \cdot f_n$.

36. The first few terms are $1, 3, 6, 10, \cdots$. This can be generated by defining $f_1 = 1$ and $f_{n+1} = f_n + n$.

37. The first few terms are $1, 2, 6, 24, \cdots$. This can be generated by defining $f_1 = 1$ and $f_{n+1} = (n + 1) \cdot f_n$.

38. The first few terms are $1/2, 2, 1/2, 2, \cdots$. This can be generated by defining $f_1 = 1/2$ and $f_{n+1} = 1/f_n$.

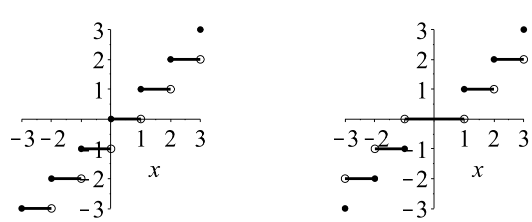
39. The first few terms are $1, 3, 6, 10, \cdots$. This can be generated by defining $f_1 = 1$ and $f_{n+1} = f_n + n$. (See Exercise 36.)

40. The first few terms are $1, 5, 17, 53, 161, \cdots$. This can be generated by defining $f_1 = 1$ and $f_{n+1} = 3 \cdot f_{n-1} + 2$.

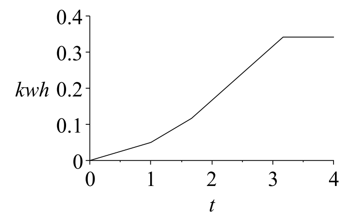
41. The function Int can be defined in terms of the floor function as follows.

$$\text{Int}(x) = \begin{cases} \lfloor x \rfloor & \text{if} \quad x \geq 0 \text{ or } x \in \mathbb{Z} \\ \lfloor x + 1 \rfloor & \text{if} \quad x < 0 \text{ and } x \notin \mathbb{Z} \end{cases}$$

The graph of the floor function is on the left, the graph of Int is on the right.



42. Measuring time t in hours, for $0 < t < 1$, a 50 watt bulb uses $50 \cdot t$ watts of energy. In kilowatt hours this is, $E(t) = 50 \cdot t/1000$ kwh. At the end of one hour, $E(t) = 50/1000 = 1/20$. From $t = 1$ to $t = 1 + 40/60 = 5/3$, $E(t) = 1/20 + 100 \cdot (t - 1)/1000$, and $E(5/3) = 7/60$. From $t = 5/3$ to $t = 5/3 + 90/60 = 19/6$, $E(t) = 7/60 + 150 \cdot (t - 5/3)/1000$, and $E(19/6) = 41/120$. From $t = 19/6$ to $t = 4$, $E(t) = 41/120$. See the graph on the right.

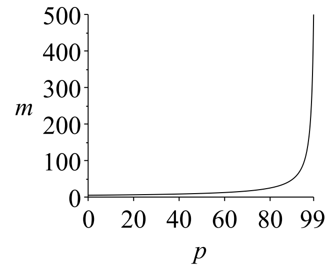


43. A monthly payment of m dollars for n years is a total payment of $m \cdot (12n)$ dollars. If the loan is P dollars, then the interest for n years is $I(P, m, n) = 12mn - P$.

44. Initially, there is 1% salt or $0.01 \times 500 = 5$ kg. Since the amount of salt does not change, when the water (M kg) makes up 98% of the solution, $0.02 \times M = 5$, and $M = 5/0.02 = 250$ kg. Evidently, the formula for $m(p)$ is

$$m(p) = 5/(1 - 0.01p)\,.$$

See the graph on the right.

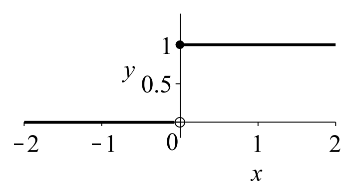


45. For c_n,

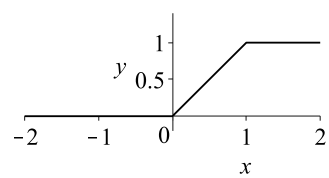
$$c_0 = 1, c_1 = c_0c_0 = 1, c_2 = c_0c_1 + c_1c_0 = 2, c_3 = c_0c_2 + c_1c_1 + c_2c_0 = 5,$$
$$c_4 = c_0c_3 + c_1c_2 + c_2c_1 + c_3c_0 = 14, c_5 = c_0c_4 + c_1c_3 + c_2c_2 + c_3c_1 + c_4c_0 = 42,$$
$$c_6 = 2(c_0c_5 + c_1c_4 + c_2c_3) = 132, c_7 = 2(c_0c_6 + c_1c_5 + c_2c_4) + c_3c_3 = 429,$$
$$c_8 = 2(c_0c_7 + c_1c_6 + c_2c_5 + c_3c_4) = 1430$$

The calculations for C_n are easier, and yield the same values. (Verify.)

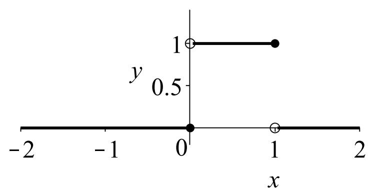
46. $H(x) = 0 \cdot (x < 0) + (x \geq 0)$, how easy was that! See its graph on the right.



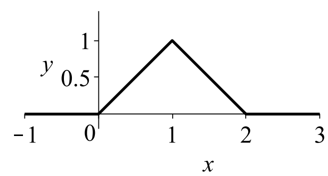
47. $R(x) = 0 \cdot (x \leq 0) + x \cdot (0 < x \leq 1) + (x > 1)$. See its graph on the right.



48. $r(x) = 0 \cdot (x \leq 0) + (0 < x \leq 1) + 0 \cdot (x > 1)$. See its graph on the right.

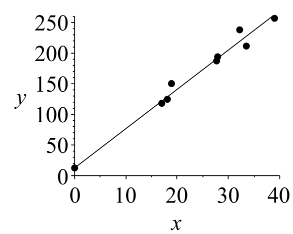


49. $b(x) = 0 \cdot (x \leq 0) + x \cdot (0 < x \leq 1) + (2-x) \cdot (2 < x \leq 2) + 0 \cdot (x > 2)$. See its graph on the right.

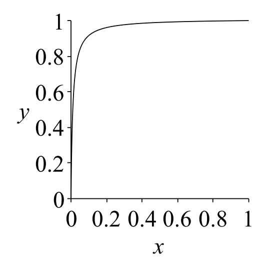


CALCULATOR/COMPUTER EXERCISES

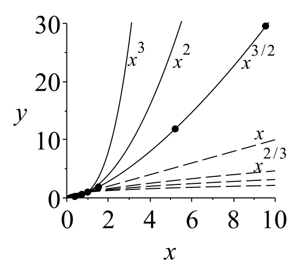
50. Using the regression line formula (see Example 11) with $(x_0, y_0) = (0, 12.5)$, we find that the slope of the line is $m = 6.42053$ and $f(x) = 6.42053x + 12.5$. See the scatter plot and the line on the right.



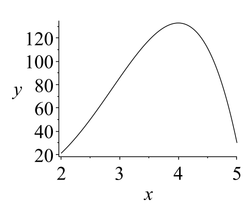
51. The graph of the probability curve $y = P(x)$ is displayed on the right. It indicates that the test should not be used if the disease occurs only rarely in the population, say under 1%. Since $P(0.01) = 0.4975$, if the population is large, then about half of the tests would be false positive.



52. The data and graphs of $y = f_q(x)$ are plotted on the right. f_q with $q = 3/2$ fits the data very well. The data falls on a curve that bends upward. This implies that $q > 1$ since $y = x^q$ bends upward for such q. The points (5.502,11.86) and (9.539,29.547) show that $q < 2$.



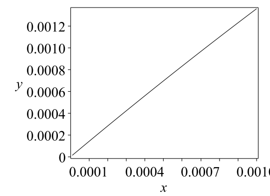
53. (a) The graph of f on the right indicates that it attains a maximum value at or near $x_0 = 4$.



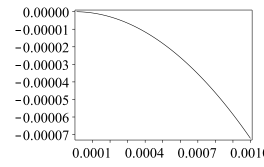
(b) The table below contains the values of $F(4,h)$ and $F(3.99,h)$

h	10^{-3}	10^{-4}	10^{-5}
$F(4,h)$	-7.2×10^{-5}	-7.2×10^{-7}	-7.2×10^{-9}
$F(3.99,h)$	0.00136	0.000143	0.0000141

(c) The graph of $h\mapsto F(3.99,h)$, displayed on the right, indicates that $F(3.99,h)\approx mx$ for $m\approx 1.4$.



(d) The graph of $h\mapsto F(4,h)$, displayed on the right, indicates that $F(4,h)\approx Ax^2$ for $A\approx -72$.



54. The calculations are displayed below.

$1.5^2=2.25>2\Rightarrow I_2=[1,1.5]$, $1.25^2=1.5625<2\Rightarrow I_3=[1.25,1.5]$
$m_3^2=1.89<2\Rightarrow I_4=[1.375,1.5]$, $m_4^2=2.066>2\Rightarrow I_5=[1.375,1.4375]$
$m_5^2=1.978<2\Rightarrow I_6=[1.40625,1.4375]$, $m_6^2=2.022>2\Rightarrow I_7=[1.40625,1.421875]$
$m_7^2=1.9996<2\Rightarrow I_8=[1.4140625,1.421875]$

Observe that the width of the n^{th} subinterval I_n is $1/2^{n-1}$ so the length of I_8 is $1/2^7=0.0078125$. This is twice the maximum error that can occur if $m_8=1.41796875$ is used to approximate $\sqrt{2}$.

55. (a) Observe that $(x_n-\sqrt{c}\,)^2=x_n^2+c-2x_n\sqrt{c}$ implying that

$$\frac{(x_n-\sqrt{c}\,)^2}{2x_n}=\frac{x_n^2+c}{2x_n}-\sqrt{c}=x_{n+1}-\sqrt{c}\,.$$

Since the left side is positive, so it the right side, and $|x_{n+1}-\sqrt{c}\,|=(x_n-\sqrt{c}\,)^2/(2x_n)$.

(b) x_n approximates $\sqrt{c}$ to k decimal places when $|x_n-\sqrt{c}\,|<5\cdot 10^{-(k+1)}$. Assuming this is the case, and using the identity in part a along with the fact that $x_n>1$ (why?),

$$|x_{n+1}-\sqrt{c}\,|=|x_n-\sqrt{c}\,|^2/(2|x_n|)<\tfrac{1}{2}\cdot 5^2\cdot 10^{-2(k+1)}<5\cdot 10^{-(2k+1)}\,.$$

(c) Using the recursion formula, $c=3.75$, and $x_0=3/2$, yields the sequence

$$1.5, 2.0, 1.9375, 1.936491936, 1.936491673, 1.936491673, 1.936491673, \cdots$$

It stabilized quickly to 1.936491673, which is correct to nine decimal places.

56. (a) The following table contains the requested data.

n	0	1	2	3	4	5	6	7	8	9	10
s_n	1	2	5	12	29	70	169	408	985	2378	5741
d_n	1	3	7	17	41	99	239	577	1393	3363	8119
r_n	1	1.5	1.4	$1.41\overline{6}$	1.4138	1.4143	1.414201	1.414216	1.414213	1.414214	1.414214

(b) This can be verified by calculating $d_n^2 - 2s_n^2$ for $n = 0, 1, \ldots, 10$. However, it can easily be established for all n by the following induction argument.

It is true if $n = 0$, because $d_0^2 - 2s_0^2 = 1 - 2 = -1 = (-1)^{0-1}$.

Let $n \geq 1$, and assume that $d_{n-1}^2 - 2s_{n-1}^2 = (-1)^{n-1-1}$. Using the recursion formulas, $d_n^2 = 4s_{n-1}^2 + 4s_{n-1}d_{n-1} + d_{n-1}^2$ and $s_n^2 = s_{n-1}^2 + 2s_{n-1}d_{n-1} + d_{n-1}^2$. Therefore,

$$d_n^2 - 2s_n^2 = 2s_{n-1}^2 - d_{n-1}^2 = -(d_{n-1}^2 - 2s_{n-1}^2) = -(-1)^{n-1-1} = (-1)^{n-1} .$$

(c) See the table in part a for r_{10}. Using the identity in part b,

$$|r_n^2 - 2| = \left| \frac{d_n^2}{s_n^2} - 2 \right| = \left| \frac{d_n^2 - 2s_n^2}{s_n^2} \right| = \frac{1}{s_n^2} .$$

Since $s_n \geq 2^n$, this shows that $|r_n^2 - 2| < 1/4^n$, which can be made as small as desired by choosing n sufficiently large.

57. The following table contains the requested data.

n	1	2	3	4	5	6	7	8
q_n	$1/\sqrt{2}$	0.923875	0.980785	0.995180	0.998795	0.999700	0.999925	0.999980
Q_n	$1/\sqrt{2}$	0.653280	0.640728	0.637640	0.636872	0.636680	0.636638	0.636625
p_n	$2\sqrt{2}$	3.06148	3.12145	3.13657	3.14034	3.14130	3.14152	3.14159

58. The following table contains the requested data.

n	0	1	2	3
a_n	$\sqrt{2}$	1.015051765	1.000027899	1.000000000
b_n	0	0.8408964155	0.9993269545	0.9999999945
p_n	$2+\sqrt{2}$	3.142606754	3.141592665	3.141592658

1.5 Combining Functions

Problems for Practice

1. $(F + G)(x) = F(x) + G(x) = x^2 + 5 + (x + 1)/(x - 1)$
$= (x^3 - x^2 + 6x - 4)/(x - 1)$

2. $(F - 3H)(x) = F(x) - 3H(x) = x^2 + 5 - 3(2x - 5) = x^2 - 6x + 20$

3. $(G \circ H)(x) = G(H(x)) = G(2x - 5) = (2x - 5 + 1)/(2x - 5 - 1)$
$$= (x - 2)/(x - 3)$$

4. $(H \circ G)(x) = H(G(x)) = H((x + 1)/(x - 1)) = 2(x + 1)/(x - 1) - 5$
$$= -(3x - 7)/(x - 1)$$

5. $(H \cdot F - H \circ F)(x) = H(x) \cdot F(x) - H(F(x)) = (2x-5)(x^2-5) - H(x^2+5)$
$$= (2x - 5)(x^2 - 5) - (2(x^2 - 5) - 5) = 2x^3 - 7x^2 + 10x - 30$$

6. $(G/F)(x) = G(x)/F(x) = (x + 1)/((x - 1)(x^2 - 5))$
$$= (x + 1)/(x^3 - x^2 + 5x - 5)$$

7. $(F \circ G \circ H)(x) = F(G(H(x))) = F(G(2x-5)) = F((2x-5+1)/(2x-5-1))$
$$= (x - 2)^2/(x - 3)^2 + 5 = (6x^2 - 34x + 49)/(x + 3)^2$$

8. $(H \circ F - F \circ H)(x) = H(F(x)) - F(H(x)) = H(x^2 + 5) - F(2x - 5)$
$$= 2(x^2 + 5) - 5 - ((2x - 5)^2 + 5) = -2x^2 + 20x - 25$$

9. $(G \circ (1/G))(x) = G(1/G(x)) = G((x - 1)/(x + 1))$
$$= ((x - 1)/(x + 1) + 1)/((x - 1)/(x + 1) - 1) = -x$$

10. $(H \circ H \circ H - H \circ H)(x) = H(H(H(x))) - H(H(x))$
$$= H(2(2x - 5) - 5) - (2(2x - 5) - 5) = H(4x - 15) - (4x - 15)$$
$$= 2(4x - 15) - 5 - 4x + 15 = 4x - 20$$

11. $h(x) = (g \circ f)(x)$ where $f(x) = x - 2$ and $g(x) = x^2$.

12. $h(x) = (g \circ f)(x)$ where $f(x) = 2x$ and $g(x) = x + 7$.

13. $h(x) = (g \circ f)(x)$ where $f(x) = x^3 + 3x$ and $g(x) = x^4$.

14. $h(x) = (g \circ f)(x)$ where $f(x) = \sqrt{x}$ and $g(x) = 3/x$.

15. Since $h(x) = 3(x + 1)^2 + 1$, $h(x) = (g \circ f)(x)$ where $g(x) = 3x^2 + 1$.

16. Since $h(x) = (x - 1)^2 + 2x + 3 = (x - 1)^2 + 2(x - 1) + 5$, $h(x) = (g \circ f)(x)$ where $g(x) = x^2 + 2x + 5$.

17. Since $h(x) = (x^2 + 1)/((x^2 + 1)^2 + 2)$, $h(x) = (g \circ f)(x)$ where $g(x) = x/(x^2 + 2)$.

18. Since $h(x) = 2(\sqrt{x}\,)^4 + (\sqrt{x}\,)^2 - (\sqrt{x}\,)^{1/3} + 1$, $h(x) = (g \circ f)(x)$ where $g(x) = 2x^4 + x^2 - x^{1/3} + 1$.

19. $(f \circ g)(1/8) = f(g(1/8)) = f(2) = 3$

20. $(g \circ f)(2) = g(f(2)) = g(3) = 3^{-1/3} = 3^{2/3}/3$

21. $f^2(11) \cdot g^3(54) = f(11)^2 \cdot g(54)^3 = 27 \cdot (1/54) = 1/2$

22. $(g \circ g)(512) = g(g(2^9)) = g(2^{-3}) = 2$

23. $x^2 + 4x - 5 = (x + 5)(x - 1)$

24. Since $x = -1$ is a root, $x+1$ is a factor. By long division, the other factor is $x^2 - 4$ (verify) so $x^3 + x^2 - 4x - 4 = (x + 1)(x + 2)(x - 2)$.

25. Both $x = 2$ and $x = -2$ are roots, so $x^2 - 4$ is a factor. Long division yields the factor x^2+2x+2 which is irreducible (complex roots). Therefore, $x^4 + 2x^3 - 2x^2 - 8x - 8 = (x - 2)(x + 2)(x^2 + 2x + 2)$.

26. Since $x^4 + 3x^2 + 2 = (x^2 + 1)(x^2 + 2)$ and both quadratics are irreducible (complex roots), this is the factorization into irreducibles of degree 2.

27. As s increases, the values $f(s)$ increase steadily from 1 to as large a number as desired. Therefore, f is both one-to-one and onto. It is invertible. Since $t = (f^{-1}(t))^2 + 1$, $t - 1 = (f^{-1}(t))^2$ and, because $f^{-1}(t)$ is not negative, $f^{-1}(t) = \sqrt{t - 1}$.

28. f is not onto because $f(s) \geq 0$ for all s. f is also not one-to-one because, for example, $f(0) = f(2) = 1$.

29. As s increases, the values $f(s)$ increase steadily from 0 to 2. Therefore, f is both one-to-one and onto. It is invertible. Since $t = s^2+s$, $s^2+s-t = 0$ and $s = (-1 \pm \sqrt{1 + 4t^2}\,)/2$. Since $s \geq 0$, the plus sign yields the formula for the values of the inverse function: $f^{-1}(t) = (-1 + \sqrt{1 + 4t^2}\,)/2$.

30. As s increases, the values $f(s)$ increase steadily from 1 to as large a number as desired. Therefore, f is both one-to-one and onto. It is invertible. Since $t = (f^{-1}(t))^4 + 1$, $t - 1 = (f^{-1}(t))^4$ and, because $f^{-1}(t)$ is not negative, $f^{-1}(t) = (t - 1)^{1/4}$.

31. As s increases from -2 to 5, the values $f(s)$ increase steadily from -35 to 98. Therefore, f is both one-to-one and onto. It is invertible. Since $t = (f^{-1}(t))^3 - 27$, $t + 27 = (f^{-1}(t))^3$ and $f^{-1}(t) = (t + 27)^{1/3}$.

32. f is neither one-to-one, nor is it onto. Observe, for example, that $f(0) = f(2^{1/3}) = 0$. Moreover, examination of the graph of f reveals that its values $f(s)$ are never less than -1.

33. Examination of the graph of f will show that as s increases from 4, its values $f(s)$ decrease steadily from 16/15 towards 1. Therefore, f is both one-to-one and onto. It is invertible. Since $t = s^2/(s^2-1)$, $t(s^2-1) = s^2$ so $(t - 1)s^2 = t$ and $s = \sqrt{t/(t - 1)}$. (Take the positive square root because s is not negative.) Therefore, $f^{-1}(t) = \sqrt{t/(t - 1)}$.

34. Examination of the graph of f will show that as s increases from 0 to 1, its values $f(s)$ increase steadily from 0 towards 1/2. Therefore, f is both

one-to-one and onto. It is invertible. Since $t = s/(s+1)$, $t(s+1) = s$ so $(t-1)s = -t$ and $s = t/(1-t)$. That is, $f^{-1}(t) = t/(1-t)$.

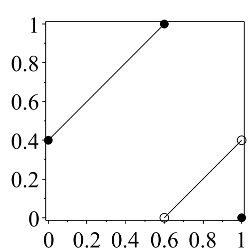
35. As s increases from 1 to 6, its values $f(s)$ increase steadily from 2 to 3. Therefore, f is both one-to-one and onto. It is invertible. Since $t = \sqrt{f^{-1}(t)+3}$, $t^2 = f^{-1}(t) + 3$ so $f^{-1}(t) = t^2 - 3$.

36. As s increases from 1 to 4, its values $f(s)$ increase steadily from 4 to 5. Therefore, f is both one-to-one and onto. It is invertible. Since $t = \sqrt{f^{-1}(t)} + 3$, $(t-3)^2 = f^{-1}(t)$ so $f^{-1}(t) = (t-3)^2$.

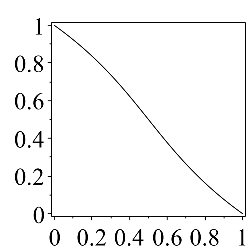
37. As s increases from 1 its values $f(s)$ decrease steadily from 1/2 towards 0. f is one-to-one, but it is not onto because 1 is in its range but it is not in its image. f does not have an inverse.

38. As s increases from 0, its values $f(s)$ decrease steadily towards 0. This makes f a one-to-one function. However, f in not onto because 1 is in its range and $f(s) < 1$ for all s in its domain.

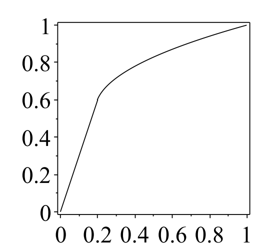
39. All six graphs represent functions but only the graphs in Figure 18 a, e, and f represent functions that are invertible. The functions for Figures b and d are onto, but not one-to-one, and the function for Figure c is one-to-one, but not onto. The graphs of the functions that are the inverses to the functions represented by Figures a, e, and f are displayed below.



Inverse for 18 a



Inverse for 18 e



Inverse for 18 f

40. $g(x) = f(x+2)$. This implies that if the the graph of f is translated 2 units to the left, then it will be the graph of g.

41. $g(x) = f(x-2)$. This implies that if the the graph of f is translated 2 units to the right, then it will be the graph of g.

42. $g(x) = f(x-3)$. This implies that if the the graph of f is translated 3 units to the right, then it will be the graph of g.

43. $g(x) = f(x-1)$. This implies that if the the graph of f is translated 1 unit to the right, then it will be the graph of g.

44. $g(x) = f(x+1) + 4$. This implies that if the graph of f is translated 1 unit to the left and 4 units up, then it will be the graph of g.

45. $g(x) = f(x+2) - 12$. This implies that if the graph of f is translated 2 unit to the left and 12 units down, then it will be the graph of g.

46. $g(x) = f(x-1) + 1$. This implies that if the graph of f is translated 1 unit to the right and 1 unit up, then it will be the graph of g.

47. $g(x) = f(x+3) - 2$. This implies that if the graph of f is translated 3 unit to the left and 2 units down, then it will be the graph of g.

48. The curve is the vertical line segment from the point (7,1) to the point (7,5).

49. Elimitate t to see that the curve is the line with Cartesian equation $y = 3(x-1) - 4$ or $y = 3x - 7$.

50. The curve is the semi-infinite horizontal line segment consisting of all points $(x, 3)$ for $x \geq 1$.

51. The curve is a parbola. Eliminating t it has the Cartesian equation $x = 12(y/2)^2 + 1$ or $x = 3y^2 + 1$. This shows that its axis of symmetry is the x-axis, its vertex is (1,0), and it opens to the right.

52. Observe that $xy = 1$. Therefore, the curve is a portion of the graph of the equation $y = 1/x$. Since $0 < x \leq 1$ and $y \geq 1$, it is the segment of the curve in the first quadrant that extends from the point (1,1) towards the positive y-axis.

Further Theory and Practice

53. The degree of the product $p \cdot q$ is the sum of the degrees: $\deg(p \cdot q) = \deg(p) + \deg(q)$. The degree of the composition is the product of the degrees: $\deg(p \circ q) = \deg(p) \cdot \deg(q)$. Because of the previous statement, $\deg(p \circ q) = \deg(q \circ p)$. The degree of $p \pm q$ will never be greater than $\deg(p)$ or $\deg(q)$. It can be less, and will be if, and only if, the degrees are the same that the terms with highest degree cancel one another.

54. If $(p \circ p)(x) = x$, then $\deg(p \circ p) = \deg(p) \cdot \deg(p) = 1$ implying that $\deg(p) = 1$ also and $p(x) = x$ or $p(x) = -x$.

55. $\deg(f) = \deg(p)^2 = n^2$, unless $p(x) = -x + a$, in which case $\deg(f) = 0$.

    If $p(r) = 0$, then $f(r) = p(r + p(r)) = p(r) = 0$ and every root of p is also a root of f. Write p as the product of the linear terms corresponding to each of its roots (real and complex, repeating if necessary): $p(x) = a(x - r_1)(x - r_2) \cdots (x - r_n)$. Then, since each root of p is a root of f, $f(x) = h(x) \cdot (x - r_1)(x - r_2) \cdots (x - r_n) = p(x) \cdot q(x)$, where $q(x) = h(x)/a$. For example, if $p(x) = x^2 - 3x - 4$, then

$$\begin{aligned} f(x) = p(x + p(x)) &= (x + x^2 - 3x - 4)^2 - 3(x + x^2 - 3x - 4) - 4 \\ &= x^4 - 4x^3 - 7x^2 + 22x + 24 = (x^2 - 3x - 4)(x^2 - x - 6) \end{aligned}$$

    where the factorization of f can be obtained by long division (verify).

56. Perform the long division to obtain

$$\frac{3x^5 + 2x^4 - x^2 + 6}{x^3 - x + 3} = 3x^2 + 2x + 3 + \frac{-8x^2 - 3x - 3}{x^3 - x + 3}.$$

57. Let $f(x) = ax + b$ and $g(x) = cx + d$ be affine functions. Then $(g \circ f)(x) = g(ax + b) = c(ax + b) + d = acx + bc + d$, which is also affine.

    The affine function f is invertible if, and only if, $a \neq 0$. Assuming this is the case, $y = af^{-1}(y) + b$ and $f^{-1}(y) = y/a - b/a$.

58. The domain of $f + c$ is the same as the domain of f, namely S. However, the range of $f + c$ is the set $\{y + c : y \in T\}$ which can be expressed as $T + c$.

    If f is one-to-one, then so is $f + c$. Moreover, if f is onto, then $f(S) = T$ and $(f + c)(S) = f(S) + c = T + c$ and $f + c$ is also onto.

    Assume f is invertible, and $(f + c)(s) = t$. Then $t = f(s) + c$ and $t - c = f(s)$ so $s = f^{-1}(t - c)$. But, $s = (f + c)^{-1}(t)$ as well, implying that $(f + c)^{-1}(t) = f^{-1}(t - c)$.

59. This is a consequence of the observations that if f and g are onto (one-to-one), then $g \circ f$ is also onto (one-to-one). Verification, which is a straightforward application of the definitions, is left to the reader.

60. The slope of ℓ' is $(s - t)/(t - s) = -1$ (assuming that $s \neq t$). This is the negative reciprocal of the slope of the line $y = x$ so ℓ and ℓ' are perpendicular.

    The equation of the line ℓ' is $y = -(x - s) + t$. It intersects the line ℓ at the point $Q = (x_0, y_0)$ where $y_0 = -(y_0 - s) + t$, so $y_0 = (s + t)/2 = x_0$ and $Q = ((s + t)/2, (s + t)/2)$. Consequently,

$$|\,\overline{PQ}\,|^2 = (s - t)^2/4 + (t - s)^2/4 = |\,\overline{QP'}\,|^2.$$

61. Since $h(x) = (x + 1)^2 + 2$, $h(x) = (g \circ f)(x)$ where $g(x) = x^2 + 2$.

62. Since $h(x) = (2x + 10)/(2x - 10) = (2x + 3 + 7)/(2x + 3 - 13)$, $h(x) = (g \circ f)(x)$ where $g(x) = (x + 7)/(x - 13)$.

63. Since $h(x) = 2(x^2 - 9) + 18$, $h(x) = (g \circ f)(x)$ where $g(x) = 2x + 18$.

64. Since $h(x) = 1/((x - 1)/x^2)$, $h(x) = (g \circ f)(x)$ where $g(x) = 1/x$.

65. Since $h(x) = (x - 4)^2 + 2$, $h(x) = (g \circ f)(x)$ where $f(x) = x - 4$.

66. Since $h(x) = (x^2+1-1)/(x^2+1+1)$, $h(x) = (g \circ f)(x)$ where $f(x) = x^2+1$.

67. Since $h(x) = ((x^2 - 1)^{1/3})^3 + 1$, $h(x) = (g \circ f)(x)$ where $f(x) = (x^2 - 1)^{1/3}$.

68. Since $h(x) = 4(2x - 5/4) + 5$, $h(x) = (g \circ f)(x)$ where $f(x) = 2x - 5/4$.

69. $f(x) = (x-3)^2$

70. $f(x) = (x-2)^3 + 2(x-2) - 3$

71. $f(x) = (-x^3+1)/(x^2+1)$

72. $f(x) = -(-x+1)/(x^4+1)$

73. If $f(x) = x^p$ and $f \circ f = f \cdot f$, then $(x^p)^p = x^p \cdot x^p$ for all x. Consequently, $x^{(p^2)} = x^{2p}$ and $x^{(p^2-2p)} = 1$ for all x. That is, either $p = 0$ or $p = 2$.

74. The figure illustrates the fact that $f(f(x)) = x$. That is, $f = f^{-1}$.

75. Let $\phi = p$. Then $\phi(f(x)) = \phi(x^5) = p(x^5) = \pi = p(x)$ for all x. That is, $\phi \circ f = p$.

    Let $\psi(x) = \pi^{1/5}$ for all x. Then $f(\psi(x)) = f(\pi^{1/5}) = (\pi^{1/5})^5 = \pi = p(x)$. That is, $f \circ \psi = p$.

    Let $\mu(x) = x^{1/5} + \pi$ for all x. Then $\mu(f(x)) = \mu(x^5) = (x^5)^{1/5} + \pi = x + \pi = I(x) + p(x) = (I+p)(x)$. That is, $\mu \circ f = I + p$.

    Let $\lambda(x) = (x+\pi)^{1/5}$ for all x. Then $f(\lambda(x)) = f((x+\pi)^{1/5}) = x + \pi = (I+p)(x)$. That is, $f \circ \lambda = I + p$.

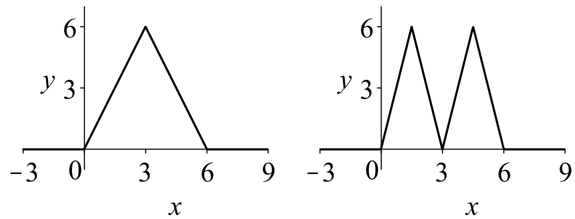
76. Let $f(x) = (ax+b)/(cx+d)$ and $g(x) = (\alpha x + \beta)/(\gamma x + \delta)$ with $ad - bc = \alpha\delta - \beta\gamma = 1$. Then

$$(g \circ f)(x) = g(f(x)) = \frac{\alpha \cdot \left(\frac{ax+b}{cx+d}\right) + \beta}{\gamma \cdot \left(\frac{ax+b}{cx+d}\right) + \delta} = \frac{\alpha(ax+b) + \beta(cx+d)}{\gamma(ax+b) + \delta(cx+d)}$$
$$= \frac{(\alpha a + \beta c)x + \alpha b + \beta d}{(\gamma a + \delta c)x + \gamma b + \delta d} = \frac{Ax+B}{Cx+D}\,.$$

    Thus $g \circ f$ has the correct *form* to be in $\mathcal{T}$. To complete the verification it must be shown that $AD - BC = 1$. This is a straightforward calculation. In fact $AD - BC = (ad - bc)(\alpha\delta - \beta\gamma)$ (verify).

    Let's calculate a formula for what must be the inverse to f. From $y = (ax+b)/(cx+d)$ we obtain $y(cx+d) = ax+b$ and $(cy-a)x = -yd+b$. Therefore, $x = (-dy+b)/(cy-a) = (dy-b)/(-cy+a) = f^{-1}(y)$ and f^{-1} has the correct *form* to be in $\mathcal{T}$. The fact that $da - (-b)(-c) = ad - bc = 1$ confirms that $f^{-1} \in \mathcal{T}$. Finally, it should also be confirmed that $f(f^{-1}(y)) = y$ for *all* y and $f^{-1}(f(x)) = x$ for *all* x. This calculation is also left to the reader.

77. The graphs of f and $f \circ f$ are displayed below. The formula for $f \circ f$ can be discerned from the plot on the right.



78. If $f(x) = mx + b$, then $r(x) = m$ for all x.

    If $f(x) = ax^2 + bx + c$, then $r(x) = a \cdot \frac{(x+h/2)^2 - (x-h/2)^2}{h} + b = 2ax + b$.

    If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, then the leading terms for the polynomial $r(x)$ derive from

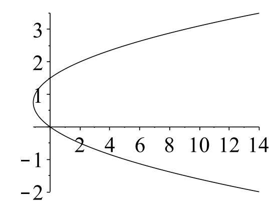
    $$a_n \cdot \frac{(x+h/2)^n - (x-h/2)^n}{h} = a_n(nx^{n-1}/2 + nx^{n-1}/2) + (\text{lower order terms})$$

    This yields $r(x) = na_n x^{n-1} + \cdots + a_1$, which is of degree $n - 1$.

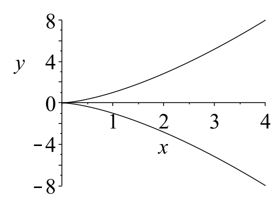
79. Let $G(x) = x^p$ and $F(x) = px$. Then $\log_{10}(x^p) = (\log_{10} \circ G)(x)$ and $p \cdot \log_{10}(x) = (F \circ \log_{10})(x)$.

    Let $H(x) = \log_{10}(p) + x$ for all x. Then $(\log_{10} \circ F)(x) = \log_{10}(px)$ and $(H \circ \log_{10})(x) = H(\log_{10}(x)) = \log_{10}(p) + \log_{10}(x)$.

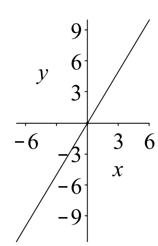
80. This is a parabola. Eliminate t and the curve has the Cartesian equation $x = 2(y-1)^2 + y - 2$ or $x + 1 = (y-1)(2y-1)$. This implies that the parabola opens to the right. See the picture, which displays the portion of the curve corresponding to $-3 \le t \le 2.5$.



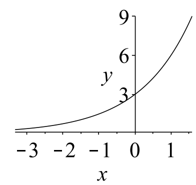
81. Eliminate t and the curve has the Cartesian equation $x = y^{2/3}$. Note that y can be chosen arbitrarily and $x \ge 0$ for all t. See the picture, which displays the portion of the curve corresponding to $-2 \le t \le 2$.



82. Since $\log_{10}(t) = y/5$ the points lie on the graph of the Cartesian equation $x = 3 \cdot (y/5)$ or $y = 5x/3$. The graph consists of all points on this line. See the picture which displays the points corresponding to $0.005 \le t \le 100$.



83. Since $\log_{10}(t) = \log_{10}(2) \cdot x$, $t = 10^{\log_{10}(2)\cdot x} = 2^x$. Therefore, the curve is the graph of the Cartesian equation $y = 3 \cdot 2^x$. See the picture, which displays the points corresponding to $0.1 \le t \le 3$.



84. Eliminate t from the equations $x = (1-t)\cdot x_1 + t\cdot x_2$ and $y = (1-t)\cdot y_1 + t\cdot y_2$ (solve the first equation for t and substitute into the second) to obtain the following equation.

$$y = \frac{y_1 - y_2}{x_1 - x_2} \cdot (x - x_1) + y_1$$

This is the equation of the line joining (x_1, y_1) to (x_2, y_2). When $t = 0$ the parametric equations yield the point (x_1, y_1) and when $t = 1$ they yield the point (x_2, y_2).

85. f is even when $f(-x) = f(x)$ for all x. Thus f is even if, and only if, $f \circ S = f$.

g is odd when $g(-x) = -g(x)$ for all x. Thus g is odd if, and only if, $g \circ S = S \circ g$.

86. (a) i. Given *any* function f, defined on all of $\mathbb{R}$, the function $g(x) = (f(x) + f(-x))/2$ is even. Indeed, $g(-x) = (f(-x) + f(x))/2 = g(x)$.

ii. Given *any* function f, defined on all of $\mathbb{R}$, the function $k(x) = (f(x) - f(-x))/2$ is odd. Indeed, $k(-x) = (f(-x) - f(x))/2 = -(f(x) - f(-x))/2 = -k(x)$.

Since $f = g + k$, this shows that every function defined on all of $\mathbb{R}$ can be written as the sum of an even function and an odd function. If f is a polynomial, then g and k will be polynomials.

(b) Writing $p = q + r$ as indicated in part a, $q(x) = (p(x) + p(-x))/2$ has only even powers of x since every odd power term in $p(x)$: $a_k x^k$, is cancelled by the corresponding odd powered term in $p(-x)$: $a_k(-x)^k = -a_k x^k$.

Similarly, $r(x) = (p(x) - p(-x))/2$ has only odd powers because every even power term in $p(x)$: $a_k x^k$, is cancelled by the corresponding even powered term in $p(-x)$: $a_k(-x)^k = a_k x^k$.

(c) If p is even, then by what is shown in part b, $r(x) = 0$ and $p(x) = q(x)$ so it contains only even terms. Similarly, if p is odd, then $q(x) = 0$ and $p(x) = r(x)$ which contains only odd terms.

(d) If p is even, then a typical nonzero and nonconstant term is of the form $a_{2k}x^{2k} = a_{2k}(x^2)^k$. Consequently, $p(x) = a_{2n}(x^2)^n + \cdots +$

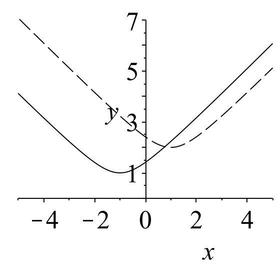
$a_2(x^2) + a_0 = s(x^2)$ where s is the polynomial $s(x) = a_{2n}x^n + \cdots + a_2x + a_0$.

(e) Let p be an odd polynomial. Then $p(0) = p(-0) = -p(0)$ so $2p(0) = 0$ implying that $p(0) = 0$. (A similar argument shows that $f(0) = 0$ for *any* odd function f.)

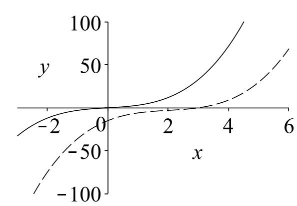
Since every summand in p is nonconstant and odd powered we may factor an x out of each one to write $p(x) = x \cdot t(x)$ where t has only even powered terms and possibly a constant term as well. This implies that t is an even polynomial.

CALCULATOR/COMPUTER EXERCISES

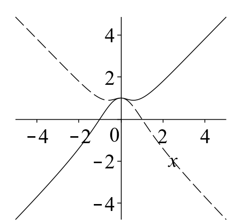
87. Let $f(x) = \sqrt{x^2 + 2x + 2}$ and plot $y = f(x)$ and $y = f(x-2) + 1$. See the picture on the right where the translated graph is the dashed curve.



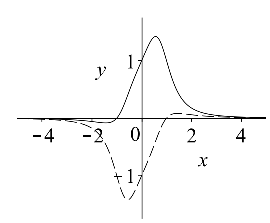
88. Let $f(x) = x^3 + 2x$ and plot $y = f(x)$ and $y = f(x-2) - 3$. See the picture on the right where the translated graph is the dashed curve.



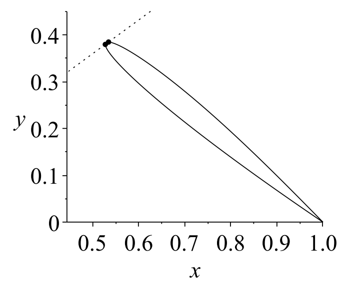
89. Let $f(x) = (x^3+1)/(x^2+1)$ and plot $y = f(x)$ and $y = f(-x)$. See the picture on the right where the reflected graph is the dashed curve.

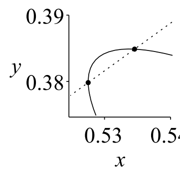


90. Let $f(x) = (x+1)/(x^4+1)$ and plot $y = f(x)$ and $y = -f(-x)$. See the picture on the right where the reflected graph is the dashed curve.

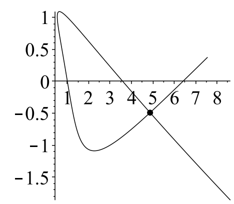


91. The curve and line $y = 18x/25$ are plotted on the right. The line intersects the curve in two points. Using *Maple*'s *fsolve* procedure the t values for the intersection are $t = -0.63068$ and $t = -0.57399$. See the close-up below.

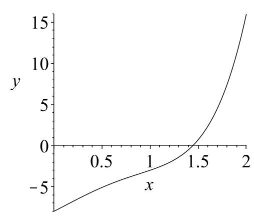




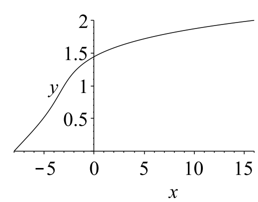
92. The curve is sketched on the right, along with the double point. Using *Maple*'s *fsolve* procedure the t values for the double point are $t = -1.52403$ and $t = 1.26993$.



93. The graph of f appears below on the left. Its inverse is drawn on the right. It was obtained by plotting the parametrized curve $y = t$, $x = f(t)$.

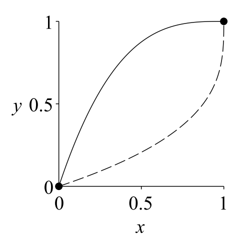


The graph of f.



The graph of f^{-1}.

94. The graph of f and its inverse appear in the plot on the right. The inverse's graph is the dashed curve. It was obtained by plotting the parametrized curve $y = t$, $x = f(t)$.



1.6 Trigonometry

Problems for Practice

1. Consult the standard 30-60-90 triangle to see that $\sin(\pi/6) = 1/2$ and $\cos(\pi/6) = \sqrt{3}/2$. Values for the other four functions can be given in

terms of these: $\tan(\pi/6) = \sqrt{3}/3$, $\cot(\pi/6) = 3/\sqrt{3} = \sqrt{3}$, $\csc(\pi/6) = 2$, and $\sec(\pi/6) = 2/\sqrt{3} = 2\sqrt{3}/3$.

2. Consult the standard 45-45-90 triangle to see that $\sin(\pi/4) = 1/\sqrt{2} = \sqrt{2}/2$ and $\cos(\pi/4) = \sqrt{2}/2$. Values for the other four functions can be given in terms of these: $\tan(\pi/4) = 1$, $\cot(\pi/4) = 1$, $\csc(\pi/4) = \sqrt{2}$, and $\sec(\pi/4) = \sqrt{2}$.

3. The angle $2\pi/3$ puts the radial line into the second quadrant. With a standard 30-60-90 triangle in the correct position we have $\sin(2\pi/3) = \sqrt{3}/2$ and $\cos(2\pi/3) = -1/2$. Values for the other four functions can be given in terms of these: $\tan(2\pi/3) = -\sqrt{3}$, $\cot(2\pi/3) = -\sqrt{3}/3$, $\csc(2\pi/3) = 2\sqrt{3}/3$, and $\sec(2\pi/3) = -2$.

4. The angle $4\pi/3$ puts the radial line into the third quadrant. With a standard 30-60-90 triangle in the correct position we have $\sin(4\pi/3) = -\sqrt{3}/2$ and $\cos(4\pi/3) = -1/2$. Values for the other four functions can be given in terms of these: $\tan(4\pi/3) = \sqrt{3}$, $\cot(4\pi/3) = \sqrt{3}/3$, $\csc(4\pi/3) = -2\sqrt{3}/3$, and $\sec(4\pi/3) = -2$.

5. Using a 30-60-90 triangle, $\sin(\pi/3)\sin(\pi/6) = \sqrt{3}/2 \cdot 1/2 = \sqrt{3}/4$.

6. $\cos(0) - \cos(\pi) = 1 - (-1) = 2$

7. $\cos(\pi/6) + \cos(\pi/3) = \sqrt{3}/2 + 1/2 = (\sqrt{3} + 1)/2$

8. $\sin(\pi/4)\cos(\pi/4) = 1/\sqrt{2} \cdot 1/\sqrt{2} = 1/2$

9. $\tan(\pi/3)/\tan(\pi/6) = \sqrt{3}/(1/\sqrt{3}\,) = 3$

10. $\cos(2\pi/3)\csc(2\pi/3) = -1/2 \cdot 2/\sqrt{3} = -\sqrt{3}/3$

11. $\sin(\pi \cdot \sin(\pi/6)) = \sin(\pi/2) = 1$

12. $\sec(-\pi/3)^{\csc(-\pi/2)} = 2^{-1} = 1/2$

13. $\sin(19\pi/2)^{\cos(33\pi)} = \sin(3\pi/2)^{\cos(\pi)} = (-1)^{-1} = -1$

14. $4\tan(\pi/4) - \sin(17\pi/2) = 4 \cdot 1 - \sin(\pi/2) = 4 - 1 = 3$

15. If $\sin(\theta) = 1/3$, then $\cos(\theta) = \sqrt{1 - \sin^2(\theta)} = \sqrt{8/9} = 2\sqrt{2}/3$ (the cosine function is positive in the first quadrant).

16. If $\cos(\theta) = 3/5$, then $\sin(\theta) = \sqrt{1 - \cos^2(\theta)} = 4/5$ (the sine function is positive in the first quadrant) and $\tan(\theta) = 4/3$.

17. If $\cos(\theta) = 4/5$, then $\sin(\theta) = \sqrt{1 - \cos^2(\theta)} = 3/5$ (the sine function is positive in the first quadrant) and $\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2 \cdot (3/5) \cdot (4/5) = 24/25$.

18. If $\cos(\theta) = 3/7$, then $\sin^2(\theta/2) = (1 - \cos(\theta))/2 = (1 - 3/7)/2 = 2/7$ and $\sin(\theta/2) = \sqrt{2/7} = \sqrt{14}/7$ (the sine function is positive in the first quadrant).

19. If $\sin(\theta) = 5/13$, then $\cos(\theta) = \sqrt{1 - \sin^2(\theta)} = 12/13$ (the sine function is positive in the first quadrant) and $\cos^2(\theta/2) = (1 + \cos(\theta))/2 = 25/26$. Therefore, $\cos(\theta/2) = \sqrt{25/26} = 5\sqrt{26}/26$.

20. $\cos(\theta + \pi) = \cos(\theta)\cos(\pi) - \sin(\theta)\sin(\pi) = -\cos(\theta) = -0.1$

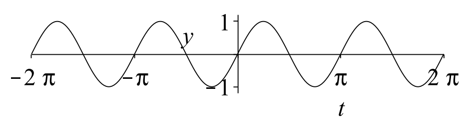
21. All six trigonometric functions are positive in the first quadrant.

22. In the second quadrant the sine and cosecant functions are positive.

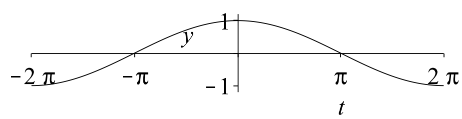
23. In the third quadrant the tangent and cotangent functions are positive.

24. In the fourth quadrant the cosine and secant functions are positive.

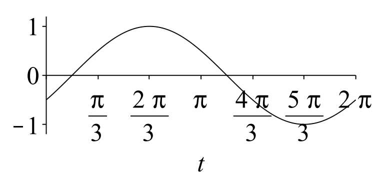
25. Since $\sin(2t)$ has period π, this is four complete sine waves.



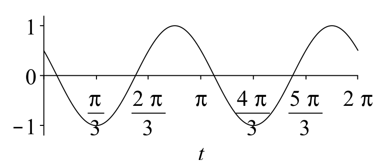
26. Since $\cos(t/2)$ has period 4π, this is one complete cosine wave.



27. Since $\sin(t - \pi/6)$ has period 2π, this is one complete sine wave shifted to the right $\pi/6$ units.



28. Since $\cos(2t + \pi/3) = \cos(2(t + \pi/6))$ has period π, this is two complete cosine waves shifted to the left $\pi/6$ units.



29. The parametrization $\phi_1(t) = \cos(t)$, $\phi_2(t) = \sin(t)$ does the job.

30. The parametrization $\phi_1(t) = \cos(t + \pi/2)$, $\phi_2(t) = \sin(t + \pi/2)$ does this job. It simplifies to $\phi_1(t) = -\sin(t)$, $\phi_2(t) = \cos(t)$. Note that it starts at the right point and, initially, x decreases and y decreases.

31. The parametrization $\phi_1(t) = \sin(t)$, $\phi_2(t) = \cos(t)$ will handle this. Note that it starts at the right place and, initially, x increases and y decreases.

Further Theory and Practice

32. This is neat.

$$\sin(0) = \sqrt{0}/2 \quad , \quad \sin(\pi/6) = \sqrt{1}/2 \quad , \quad \sin(\pi/4) = \sqrt{2}/2$$
$$\sin(\pi/3) = \sqrt{3}/2 \quad , \quad \sin(\pi/2) = \sqrt{4}/2$$

33. There is a constant k such that $s(r,\theta) = k \cdot r \cdot \theta$. k can be found by substituting $r = 1$ and $\theta = 2\pi$ to obtain $2\pi = k \cdot 1 \cdot 2\pi$. Therefore, $k = 1$, and $s(r,\theta) = r \cdot \theta$.

34. There is a constant k such that $A(r,\theta) = k \cdot r^2 \cdot \theta$. k can be found by substituting $r = 1$ and $\theta = 2\pi$ to obtain $\pi = k \cdot 1 \cdot 2\pi$. Therefore, $k = 1/2$, and $A(r,\theta) = \frac{1}{2} \cdot r^2 \cdot \theta$.

35. The second equation, $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$, is an identity, true for all values of θ. The first equation, $\sin(2\theta) = \sin(\theta)\cos(\theta)$, is true only for some values of θ.

    Using the identity, the second equation is equivalent to $\sin(2\theta) = \sin(2\theta)/2$ which is true if, and only if, $\sin(2\theta) = 0$. Thus the second equation is true whenever $2\theta = n\pi$, where n is an integer. That is, $\theta = n\pi/2$ where n can be any integer.

36. Observe that $\sin(3\theta) = \sin(\theta + 2\theta) = \sin(\theta)\cos(2\theta) + \cos(\theta)\sin(2\theta)$. Consequently,

$$\begin{aligned}\sin(3\theta) &= \sin(\theta)(\cos^2(\theta) - \sin^2(\theta)) + \cos(\theta) \cdot 2\sin(\theta)\cos(\theta) \\ &= 3\cos^2(\theta)\sin(\theta) - \sin^3(\theta) = 3(1 - \sin^2(\theta))\sin(\theta) - \sin^3(\theta) \\ &= 3\sin(\theta) - 4\sin^3(\theta)\,.\end{aligned}$$

37. Convert to sines and cosines, apply the sum identities, and then divide the top and bottom by $\cos(\theta)\cos(\phi)$.

$$\begin{aligned}\tan(\theta + \phi) = \frac{\sin(\theta + \phi)}{\cos(\theta + \phi)} &= \frac{\sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi)}{\cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)} \\ &= \frac{\tan(\theta) + \tan(\phi)}{1 - \tan(\theta)\tan(\phi)}\,.\end{aligned}$$

38. Start from the right side, applying the sum identities for cosine. The details are left to the reader (the cosine terms cancel).

$$\begin{aligned}\tfrac{1}{2}(\cos(\theta - \phi) - \cos(\theta + \phi)) &= \tfrac{1}{2}(\sin(\theta)\sin(\phi) - (-\sin(\theta)\sin(\phi))) \\ &= \sin(\theta)\sin(\phi)\end{aligned}$$

39. Start from the right side, applying the sum identities for cosine. The details are left to the reader (the sine terms cancel).

$$\begin{aligned}\tfrac{1}{2}(\cos(\theta - \phi) + \cos(\theta + \phi)) &= \tfrac{1}{2}(\cos(\theta)\cos(\phi) + \cos(\theta)\cos(\phi)) \\ &= \cos(\theta)\cos(\phi)\end{aligned}$$

40. Start from the right side, applying the sum identities for sine. The details are left to the reader (two of the sine-cosine terms cancel).

$$\begin{aligned}\tfrac{1}{2}(\sin(\theta+\phi)+\sin(\theta-\phi)) &= \tfrac{1}{2}(\sin(\theta)\cos(\phi)+\sin(\theta)\cos(\phi)) \\ &= \sin(\theta)\cos(\phi)\end{aligned}$$

41. Start from the right side, applying the sum identity for cosine.

$$\cos\left(\tfrac{\pi}{2}-\theta\right) = \cos\left(\tfrac{\pi}{2}\right)\cos(\theta)+\sin\left(\tfrac{\pi}{2}\right)\sin(\theta) = \sin(\theta)$$

42. Start from the left side, applying the sum identity for sine.

$$\sin(\theta+\pi) = \sin(\theta)\cos(\pi)+\cos(\theta)\sin(\pi) = -\sin(\theta)$$

43. Start from the left side, applying the sum identity for cosine.

$$\cos(\theta+\pi) = \cos(\theta)\cos(\pi)-\sin(\theta)\sin(\pi) = -\cos(\theta)$$

44. Convert to sines and cosines and apply the sum identities.

$$\begin{aligned}\tan(\theta+\pi) = \frac{\sin(\theta+\pi)}{\cos(\theta+\pi)} &= \frac{\sin(\theta)\cos(\pi)+\cos(\theta)\sin(\pi)}{\cos(\theta)\cos(\pi)-\sin(\theta)\sin(\pi)} \\ &= \frac{-\sin(\theta)}{-\cos(\theta)} = \tan(\theta)\,.\end{aligned}$$

45. Begin with the sine and cosine calculation.

$$\begin{aligned}\sin(7\pi/12) = \sin(\pi/3)\cos(\pi/4)+\cos(\pi/3)\sin(\pi/4) &= (\sqrt{3}/2+1/2)\sqrt{2}/2 \\ &= (\sqrt{6}+\sqrt{2}\,)/4 = \tfrac{1}{2}\sqrt{2+\sqrt{3}} \\ \cos(7\pi/12) = \cos(\pi/3)\cos(\pi/4)-\sin(\pi/3)\sin(\pi/4) &= (1/2-\sqrt{3}/2)\sqrt{2}/2 \\ &= (\sqrt{2}-\sqrt{6}\,)/4 = -\tfrac{1}{2}\sqrt{2-\sqrt{3}}\end{aligned}$$

In each case, the final answer is obtained by squaring and then simplifying the penultimate expression. The sine and cosine values can be used to find the other four. The final tangent calculation is completed by multiplying the top and bottom by $\sqrt{2+\sqrt{3}}$.

$$\begin{aligned}\tan(7\pi/12) &= \frac{\sin(7\pi/12)}{\cos(7\pi/12)} = \frac{\tfrac{1}{2}\sqrt{2+\sqrt{3}}}{-\tfrac{1}{2}\sqrt{2-\sqrt{3}}} = -2-\sqrt{3} \\ \cot(7\pi/12) &= \frac{1}{-2-\sqrt{3}} = \frac{-2+\sqrt{3}}{1} = -2+\sqrt{3} \\ \csc(7\pi/12) &= \frac{1}{\tfrac{1}{2}\sqrt{2+\sqrt{3}}} = \frac{2\sqrt{2-\sqrt{3}}}{1} = 2\sqrt{2-\sqrt{3}} \\ \sec(7\pi/12) &= \frac{1}{-\tfrac{1}{2}\sqrt{2-\sqrt{3}}} = -\frac{2\sqrt{2+\sqrt{3}}}{1} = -2\sqrt{2+\sqrt{3}}\end{aligned}$$

46. Begin with the sine and cosine calculation.

$$\sin(\pi/8) = \sin((\pi/4)/2) = \sqrt{\frac{1-\cos(\pi/4)}{2}} = \sqrt{\frac{1-\sqrt{2}/2}{2}} = \tfrac{1}{2}\sqrt{2-\sqrt{2}}$$

$$\cos(\pi/8) = \cos((\pi/4)/2) = \sqrt{\frac{1+\cos(\pi/4)}{2}} = \sqrt{\frac{1+\sqrt{2}/2}{2}} = \tfrac{1}{2}\sqrt{2+\sqrt{2}}$$

The sine and cosine values can be used to find the other four.

$$\begin{aligned}
\tan(\pi/8) &= \frac{\sin(\pi/8)}{\cos(\pi/8)} = \sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}} = \sqrt{\tfrac{1}{2}(4-4\sqrt{2}+2)} = \sqrt{3-2\sqrt{2}} \\
\cot(\pi/8) &= \frac{1}{\sqrt{3-2\sqrt{2}}} = \sqrt{3+2\sqrt{2}} \\
\csc(\pi/8) &= \frac{2}{\sqrt{2-\sqrt{2}}} = \sqrt{2(2+\sqrt{2}\,)} \\
\sec(\pi/8) &= \frac{2}{\sqrt{2+\sqrt{2}}} = \sqrt{2(2-\sqrt{2}\,)}
\end{aligned}$$

47. Substitute $\frac{\theta+\phi}{2}$ for θ and $\frac{\theta-\phi}{2}$ for ϕ in the identity in Exercise 40.

48. Substitute $\frac{\theta+\phi}{2}$ for ϕ and $\frac{\theta-\phi}{2}$ for θ in the identity in Exercise 40.

49. Substitute $\frac{\theta+\phi}{2}$ for θ and $\frac{\theta-\phi}{2}$ for ϕ in the identity in Exercise 39.

50. Substitute $\frac{\theta+\phi}{2}$ for θ and $\frac{\theta-\phi}{2}$ for ϕ in the identity in Exercise 38.

51. Let $C = \sqrt{A^2+B^2}$ and observe that the point $(B/C, A/C)$ lies on the unit circle in the xy plane. Let ϕ be an angle determined by the line from the origin to this point so $B/C = \cos(\phi)$ and $A/C = \sin(\phi)$. Then

$$\begin{aligned}
A\cos(\theta) + B\sin(\theta) &= C \cdot \left(\frac{A}{C}\cos(\theta) + \frac{B}{C}\sin(\theta)\right) \\
&= C(\sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi)) \\
&= C\sin(\theta+\phi)\,.
\end{aligned}$$

52. Name the line ℓ. Draw a unit circle centered at the point P where ℓ intersects the x-axis. Observe that ℓ intersects the circle in the upper half-plane at a point Q whose coordinates, with respect to the coordinate system with origin translated to P, are $(\cos(\phi), \sin(\phi))$. Consequently, $\tan(\phi) = \sin(\phi)/\cos(\phi)$ and $\sin(\phi)/\cos(\phi) = m$, the slope of the line. That is, $m = \tan(\phi)$.

This relationship can be extended to horizontal lines for in this case the angle ϕ could be either 0 or π and $m = \tan(\phi)$ still holds true. If the line is vertical, then m and $\tan(\phi)$ are both undefined.

ϕ is independent of b because changing b produces parallel lines with the same slope.

53. Let d be the distance from the θ vertex to the point where the vertical height line meets the horizontal line. Then $\cot(\theta) = d/h$ and $\cot(\phi) = (\ell + d)/h$. Eliminating d, $\cot(\phi) = (\ell + h\cot(\theta))/h$ or $\cot(\phi) = \ell/h + \cos(\theta)$. Consequently, $\ell/h = \cot(\phi) - \cot(\theta)$ so

$$h = \frac{\ell}{\cot(\phi) - \cot(\theta)} = \frac{\ell}{|\cot(\theta) - \cot(\phi)|}.$$

54. This is an ellipse. To see why note that $x/a = \cos(\theta)$ and $x/b = \sin(\theta)$. Therefore, $x^2/a^2 + y^2/b^2 = \cos^2(\theta) + \sin^2(\theta) = 1$.

55. This is a portion of a line. Note that $x/a + y/b = \cos^2(\theta) + \sin^2(\theta) = 1$. Observe that when $\theta = 0$, $(x, y) = (a, 0)$ and when $\theta = \pi/2$, $(x, y) = (0, b)$. The line segment joins the two intercepts.

56. Use the identity $\sec^2(\theta) - \tan^2(\theta) = 1$ to see that this is a portion of the hyperbola $x^2/a^2 - y^2/b^2 = 1$. Half of one branch is traced out as θ increases from 0 to $\pi/2$.

57. Observe that $(x/a) \cdot (y/b) = 1$ so this parametrization traces out a portion of the curve $xy = ab$.

58. The ellipse has the Cartesian equation $(x - h)^2/a^2 + (y - k)^2/b^2 = 1$. Therefore, if $(x - h)/a = \cos(\theta)$ and $(y - k)/b = \sin(\theta)$, then for each value of θ in $[0, 2\pi)$ the point (x, y) will the on the ellipse and, since every point on the ellipse can be reached in this way, $x = h + a\cos(\theta)$, $y = k + b\sin(\theta)$, $\theta \in [0, 2\pi)$ parametrizes the entire curve.

59. Since 2π is the smallest positive number such that $\sin(x + \sqrt{3} + 2\pi) = \sin(x + \sqrt{3})$ for all x, this function has period 2π.

60. Observe that $\cos(2\pi(x + 1)) = \cos(2\pi x + 2\pi) = \cos(2\pi x)$. Since 1 is the smallest positive number for which this is true for all x, this is the period.

61. The tangent function has period π. This can be shown using the sum identity:

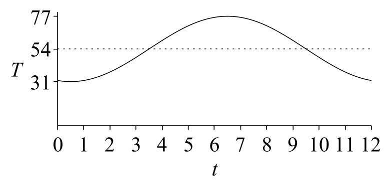
$$\tan(x + \pi) = \frac{\tan(x) + \tan(\pi)}{1 - \tan(x)\tan(\pi)} = \tan(x)$$

for all x. Examine the graph of the tangent function to see that no smaller positive number has this property.

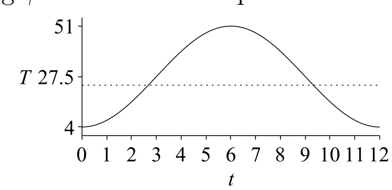
62. The sine function has period 2π and the tangent function has period π. Therefore, their sum will repeat every 2π units. This is the smallest positive number having this property because the sine function requires 2π units to cycle back to its previous values.

63. $\tan(2x)$ has period $\pi/2$ and $\sin(3x)$ has period $2\pi/3$. They will both repeat over any interval whose width is a common multiple of $\pi/2$ and $2\pi/3$. The smallest positive number that has this property is $6\pi/3 = 2\pi$.

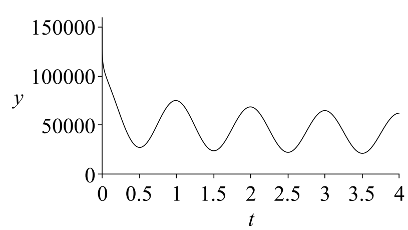
64. The constant b should be the average of 31 and 77: $b = 54$, and the amplitude A will be the common difference between the extremes and the average: $A = 23$. We will measure time t in months so the value of ω must be chosen to yield a period of 12. This implies that $12\omega = 2\pi$ so $\omega = \pi/6$. At this point $T(t) = 54 + 23\sin(\pi t/6 + \phi)$ and it remains to choose a phase angle ϕ so that the extreme values of T are attained half of the way into January and half of the way into July. This is accomplished by choosing ϕ so that the sine part of the definition attains the value -1 at $t = 1/2$: $\sin(\pi/12 + \phi) = -1$. So, for example, choose ϕ to solve the equation $\pi/12 + \phi = 3\pi/2$. That is, $\phi = 3\pi/2 - \pi/12 = 17\pi/12$. See the picture on the right.



65. The constant b should be the average of 4 and 51: $b = 27.5$, and the amplitude A will be the common difference between the extremes and the average: $A = 23.5$. We will measure time t in months so the value of ω must be chosen to yield a period of 12. This implies that $12\omega = 2\pi$ so $\omega = \pi/6$. At this point $T(t) = 27.5 + 23.5\sin(\pi t/6 + \phi)$ and it remains to choose a phase angle ϕ so that the maximum value of T is attained on July 1^{st}. This is accomplished by choosing ϕ so that the sine part of the definition attains the value 1 at $t = 6$: $\sin(\pi + \phi) = 1$. So, for example, choose ϕ to solve the equation $\pi + \phi = \pi/2$. That is, $\phi = \pi/2 - \pi = -\pi/2$. See the picture on the right.



66. The graph appears on the right. Since the y values are tending downward, y is not a periodic function of t. There are "seasonal highs" at $t = 1, 2, 3, \ldots$. These are time values t_0 such that $y(t_0)$ is larger than all $y(t)$ for t sufficiently close to t_0.

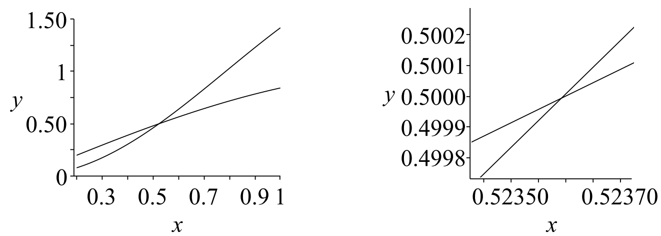


67. (a) The inscribed n-gon splits naturally into n congruent isosceles triangles, each one having a vertex angle $2\pi/n$ and equal sides of length r. Therefore, each triangle has base $2 \cdot r\sin(\pi/n)$. The sum of the n base lengths is $p(n, r)$: $p(n, r) = n \cdot 2r\sin(\pi/n)$.

(b) The area of each inscribed isosceles triangle is $\frac{1}{2} \cdot (\text{base}) \cdot (\text{height}) = r\sin(\pi/n) \cdot r\cos(\pi/n) = r^2 \sin(\pi/n)\cos(\pi/n) = \frac{1}{2}r^2\sin(2\pi/n)$. Since there are n of them, $A(n,r) = \frac{1}{2}r^2 n \sin(2\pi/n)$.

CALCULATOR/COMPUTER EXERCISES

68. The graph appears below on the left. The zoomed version is on the right.



To three decimals, $x_0 = 0.534$ and $\sin(x_0) = 0.500$. Using a half-angle identity, the equation $\sin(x) = 1 - \cos(2x)$ is equivalent to $1 - \cos(2x) = \sqrt{(1-\cos(2x))/2}$, implying that $(1-\cos(2x))^2 = (1-\cos(2x))/2$. Since $1 - \cos(2x) \neq 0$ this is equivalent to $1 - \cos(2x) = 1/2$ or $\cos(2x) = 1/2$. Therefore, $2x = \pi/3$ and $x_0 = \pi/6$. As we expect, $\sin(x_0) = 1/2$.

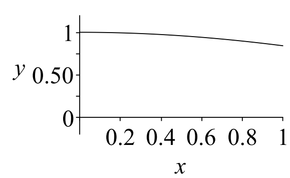
69. The following table contains the initial data, and one more (4-decimal accuracy).

n	1	2	3	4	5	6	7	8
a_n	0.8415	0.9589	0.9816	0.9896	0.9933	0.9954	0.9966	0.9974
n	9	10	11	12	13	14	15	16
a_n	0.9979	0.9983	0.9986	0.9988	0.9990	0.9992	0.9993	0.9993

The next table contains the data for $n = 10^k$, (10-decimal accuracy).

n	10^1	10^2	10^3
a_n	0.9983341665	0.9999833334	0.9999998333
n	10^4	10^5	10^6
a_n	0.9999999983	1.000000000	1.000000000

The tabulated data suggests that the limiting value of a_n as n increases is $\ell = 1$. This is supported by the graph of $y = \sin(x)/x$ displayed on the right.



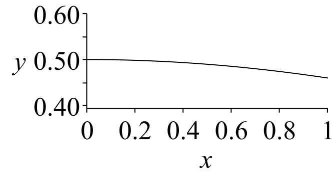
70. The following table contains the initial data, and one more (5-decimal accuracy).

n	1	2	3	4	5	6	7	8
a_n	0.45970	0.48967	0.49539	0.49740	0.49834	0.49884	0.49915	0.49935
n	9	10	11	12	13	14	15	16
a_n	0.49949	0.49958	0.49966	0.49971	0.49975	0.49979	0.49981	0.49984

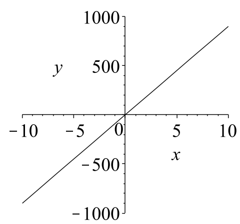
The next table contains the data for $n = 10^k$, (10-decimal accuracy).

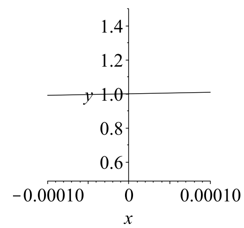
n	10^1	10^2	10^3	10^4
a_n	0.4995834722	0.4999958333	0.4999999583	0.4999999995

The tabulated data suggests that the limiting value of a_n as n increases is $\ell = 1/2$. The graph of $y = (1 - \cos(x))/x^2$ displayed on the right supports this conclusion. Observe that $y(1/n) = n^2(1 - \cos(1/n))$.



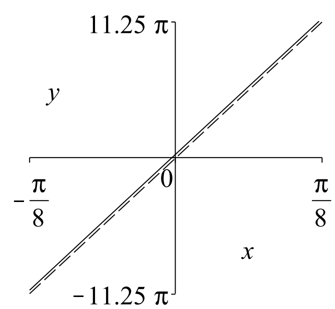
71. The graphs appear below. The zoomed version is on the right.



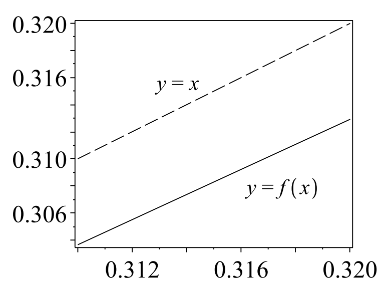


On the left we are too far away from the graph to see the oscillations due to the cosine term. On the right we are so close that the graph appears to be a straight line of slope 0.

The plot on the right gives somewhat of an idea of the behavior of f. The window is 90 times higher than it is wide so the graph of f, which is the solid line, appears to have slope 1. The dashed line is the graph of $y = 90x$ and the effect of the cosine function can be seen, if only barely. If we move closer, then the picture seems to be of two parallel lines. If we move away, even slightly, the two lines appear to coalesce into one.



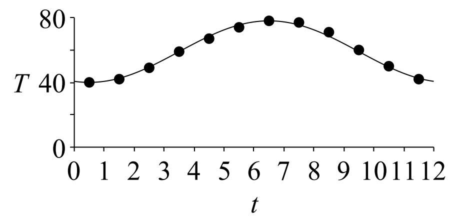
72. The error is symmetric around the origin and largest at the endpoints of such an interval. The plot on the right, which displays the graph of $y = f(x)$ as a solid line and the graph of $y = x$ as a dashed line, indicates that the error is less than 0.01 on the interval $-0.32 \leq x \leq 0.32$.



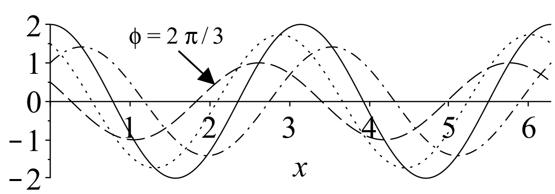
73. Assuming that the normal temperature is recorded at the midpoint of the

month we seek a temperature function $T(t) = b + A\sin(\omega t + \phi)$ with period 12 (time is measured in months) having a low of 40 in mid-January and a high of 78 in mid-July. $T(t)$ will oscillate around the average of these two extremes: $b = 59$, with amplitude equal to half their difference: $A = 19$. Choose ω so that the sine wave has period 12. That is, $12\omega = 2\pi$, implying that $\omega = \pi/6$, and $T(x) = 59 + 19\sin(\pi t/6 + \phi)$.

The phase shift ϕ is chosen so that the low is at $t = 0.5$ (mid-January). This is accomplished by choosing ϕ so that $\sin(\pi \cdot 0.5/6 + \phi) = -1$. Let ϕ be the solution to the equation $\pi \cdot 0.5/6 + \phi = 3\pi/2$. That is, $\phi = 3\pi/2 - \pi/12 = 17\pi/12$.



74. The plot below displays the four superimposed cosine waves. The solid curve is for $\phi = 0$, the dotted curve is when $\phi = \pi/3$, the dashed curve is when $\phi = 2\pi/3$, and the dash-dot curve is when $\phi = 3\pi/2$.



Judging from the graph, when $\phi = 2\pi/3$ the amplitude is approximately 1 and the phase shift is also approximately 1. Substitute $\theta = 2x + 2\pi/3$ and $\phi = 2x$ into the identity in Exercise 49 to obtain

$$\begin{aligned}\cos(2x) + \cos(2x + 2\pi/3) &= 2\cos((4x + 2\pi/3)/2)\cos((2\pi/3)/2) \\ &= 2\cos(\pi/3)\cos(2x + \pi/3) = \cos(2x + \pi/3)\,.\end{aligned}$$

The same idea can be used to derive the following general formula.

$$\begin{aligned}\cos(2x) + \cos(2x + \phi) &= 2\cos((4x + \phi)/2)\cos(\phi/2) \\ &= 2\cos(\phi/2)\cos(2x + \phi/2)\,.\end{aligned}$$

Therefore, the amplitude for the superimposed wave is $A = 2\cos(\phi/2)$ and the phase shift is $\theta = \phi/2$.

The table on the right shows the A and θ values for the waves plotted above. The last row is the wave magnitude at $x = 2$. It is maximum for $\phi = 3\pi/2$.

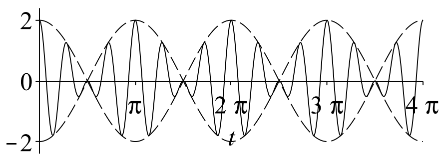
ϕ	0	$\pi/3$	$2\pi/3$	$3\pi/2$
A	2	$\sqrt{3}$	1	$-\sqrt{2}$
θ	0	$\pi/6$	$\pi/3$	$3\pi/4$
$\lvert f(2)\rvert$	1.307	0.3252	0.3287	1.410

75. Let $\theta = \nu_1 \cdot t$ and $\phi = \nu_2 \cdot t$ in the identity in Exercise 49. This yields

$$\begin{aligned}\cos(\nu_1 \cdot t) + \cos(\nu_2 \cdot t) &= 2\cos((\nu_1+\nu_2)/2 \cdot t)\cos((\nu_1-\nu_2)/2 \cdot t) \\ &= A(t)\cos(\omega t)\,.\end{aligned}$$

where $A(t) = 2\cos((\nu_1-\nu_2)/2 \cdot t)$ and $\omega = (\nu_1+\nu_2)/2$. If $\nu_1 \approx \nu_2$, then $A(t)$ is a low-frequency cosine wave and ω is about the same as the individual frequencies ν_1 and ν_2. For example, if $\nu_1 = 8$ and $\nu_2 = 6$, then $A(t) = 2\cos(t)$ and $\omega = 7$ yielding $\cos(8t) + \cos(6t) = 2\cos(t)\cos(7t)$.

See the picture on the right which plots the superimposed wave along with the time-varying envelopes $\pm 2\cos(t)$ (the dashed curves). Their frequency is 1.

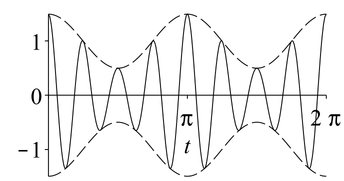


76. (a) Observe that $S(t) = A\cos(\omega t) + Am\cos(\nu t)\cos(\omega t)$. Using the identity in Exercise 39,

$$S(t) = A\cos(\omega t) + \tfrac{Am}{2}\big(\cos((\nu-\omega)t) + \cos((\nu+\omega)t)\big)\,.$$

Apparently the sidebands have frequency $\omega + \nu$ and $\omega - \nu$.

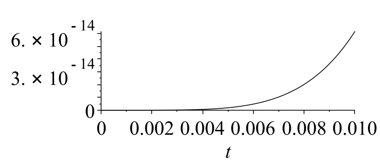
(b) The signal is plotted on the right. The envelopes are the dashed curves. We see that the carrier frequency is 8, and the envelope frequency is 2. Therefore, the sidebands have frequencies 10 and 6.

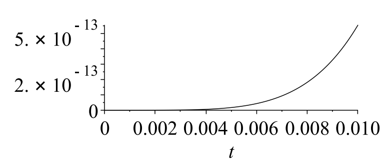


(c) The largest and smallest frequencies are $\omega + \nu_{\max}$ and $\omega - \nu_{\max}$. The bandwidth is $2\nu_{\max}$.

(d) If we assume that $\nu_{\max} = 15$ kHz, then the bandwidth is 30 kHz, implying that the minimum separation between carrier signals should also be 30 kHz.

77. Snell's inequality is illustrated on the left with the graph of the expression $\tan(t/3) + 2\sin(t/3) - t$. Cusa's inequality is illustrated on the right with the graph of the expression $t - 3\sin(t)/(2+\cos(t)$.





REVIEW EXERCISES FOR CHAPTER 1

1. All x within 6 units of -5: $(-11, 1)$.
2. All x at least 7 units away from 3: $(-\infty, -4] \cup [10, \infty)$.

3. $x(x-3)$ is 0 when $x=0$ and $x=3$. It is negative when $x \in (0,3)$, so the answer is $[0,3]$.

4. If $x < 0$, then $|x+3|/x \leq 0$, so x must be positive. Therefore, the inequality is equivalent to $|x+3| > 5x$. This implies that either $x+3 > 5x$ (which means that $3 > 4x$ so $0 < x < 3/4$) or $x+3 < -5x$ (which means that $6x < -3$, which is not possible). Therefore, the answer is $(0, 3/4)$.

5. x is either positive or it is negative. If $x > 0$, then $2x+1 < 4x$ or $1 < 2x$, yielding the interval $(1/2, \infty)$. If $x < 0$, then $2x+1 < -4x$ or $6x < -1$, yielding the interval $(-\infty, -1/6)$. The answer is $(-\infty, -1/6) \cup (1/2, \infty)$.

6. The inequality is equivalent to $x^2 - x - 6 < 0$ or $(x+2)(x-3) < 0$. Since $(x+2)(x-3)$ is 0 when $x=-2$ or $x=3$ and is negative when $x \in (-2,3)$, the answer is $(-2,3)$.

7. x is either positive or it is negative. If $x > 0$, then $x(x-1) \leq 6$ or $x^2 - x - 6 \leq 0$. That is, $(x+2)(x-3) \leq 0$, implying that $x \leq 3$. This yields the interval $(0,3]$. If $x < 0$, then $x(x-1) \geq 6$ or $x^2 - x - 6 \geq 0$. That is, $(x+2)(x-3) \geq 0$ which is true for all $x \leq -2$ yielding the interval $(-\infty, -2]$. The answer is $(-\infty, -2] \cup (0, 3]$.

8. This is equivalent to $(2x+1)^2 > (x-3)^2$ or $4x^2 + 4x + 1 > x^2 - 6x + 9$. Therefore, we solve $3x^2 + 10x - 8 > 0$. The quadratic's graph is a parabola that opens upward. Since $3x^2 + 10x - 8 = (3x-2)(x+4)$, the quadratic is 0 when $x = -4$ and $x = 2/3$ and is positive outside the open interval $(-4, 2/3)$. The answer is $(-\infty, -4) \cup (2/3, \infty)$.

9. Use the distance formula: $\sqrt{12^2 + (-5)^2} = \sqrt{144 + 25} = 13$.

10. The line meets the coordinate axes at the points $(3,0)$ and $(0,-4)$. the distance between these points is 5 units.

11. The midpoint is $((9 + (-3))/2, (-6 + (-1))/2) = (3, -7/2)$.

12. Substitute $y = 3x - 5$ into the parabola equation to obtain $3x - 5 = x^2 - 9$ or $x^2 - 3x - 4 = 0$. Therefore, the x-coordinates of the points of intersection satisfy the equation $(x+1)(x-4) = 0$ so $x = -1$ and $x = 4$. The points of intersection are $(-1, -8)$ and $(4, 7)$. The distance between them is $\sqrt{5^2 + 15^2} = 5\sqrt{10}$.

13. Complete the square: $(x^2 - 2x + 1) + (y^2 + 2y + 1) = 1 + 1$ or $(x-1)^2 + (y+1)^2 = 2$. This is a circle centered at $(1, -1)$ having radius $\sqrt{2}$. The axes of symmetry are the lines $x = 1$ and $y = -1$.

14. Complete the square in x: $y - 7/4 - 9/4 = -(x^2 - 3x + 9/4)$, or $y - 4 = -(x - 3/2)^2$. This is a parabola, opening down, with vertex $(3/2, 4)$. Its axis of symmetry is the vertical line $x = 3/2$.

15. Complete the square in x and y: $9(x^2 + 2x + 1) + 4(y^2 - 4y + 4) = 11 + 9 + 16$ or $(x+1)^2/2^2 + (y-2)^2/3^2 = 1$. This is an ellipse centered at $(-1, 2)$ having axes of symmetry $x = -1$ and $y = 2$.

16. Complete the square in y: $y^2+4y+4=4x^2+4$, or $(y+2)^2/2^2-x^2=1$. This is a hyperbola, centered at $(0,-2)$, having axes of symmetry $x=0$ and $y=-2$.

17. $y=3x+b$. When $y=0, x=2$, so $0=6+b$ and $b=-6$. The answer is $y=3x-6$.

18. The slope is $m=1/2$ so $y=(1/2)(x+1)-3$ or $y=x/2-5/2$.

19. The slope is $m=-1/(1)=-1$ so $y=-(x+1)-3$ or $y=-x-4$.

20. The slope is $m=(-5-(-1))/(3-1)=-2$ so $y=-2(x-1)-1$ or $y=-2x+1$.

21. The slope of ℓ is $m=-6$, so $y=-6(x+2)+1$.

22. The slope of ℓ is $m=-1/(-2)=1/2$, so $y=(1/2)(x+2)+1$.

23. The slope of ℓ is $m=(5-1)/(0-(-2))=2$, so $y=2(x+2)+1$.

24. The slope of ℓ is $m=(0-6)/(2-1)=-6$, so $y=-6(x-1)+6$.

25. $x/4+y/(-3)=1$

26. The slope is $m=-1$, so $y=-(x-2)+3$ or $y=-x+5$. The intercept form is $x/5+y/5=1$.

27. The slope is $m=-1/(1/2)=-2$, so $y=-2(x-1)+1$ or $y=-2x+3$. The intercept form is $x/(3/2)+y/3=1$.

28. The slope is $m=(1-5)/(3-1)=-2$, so $y=-2(x-1)+5$ or $y=-2x+7$. The intercept form is $x/(7/2)+y/7=1$.

29. Add the equations: $3x=6$, so $x=2$ and $y=3$. The lines intersect at $(2,3)$.

30. Subtract twice the second from first: $-5x=5$, so $x=-1$ and $y=4$. The lines intersect at $(-1,4)$.

31. Substitute the second equation into the first: $x^2-9=3x-5$ or $x^2-3x-4=0$. Therefore, $(x+1)(x-4)=0$ and $x=-1$ or $x=4$. If $x=-1$, then $y=-8$, and if $x=4$, then $y=7$. The intersection points are $(-1,-8)$ and $(4,7)$.

32. Substitute the first equation into the second: $x^2-3=1+2x-x^2$ or $2x^2-2x-4=0$. Therefore, $x^2-x-2=0$. That is, $(x-2)(x+1)=0$, and $x=2$ or $x=-1$. If $x=2$, then $y=1$, and if $x=-1$, then $y=-2$. The intersection points are $(2,1)$ and $(-1,-2)$.

33. The denominator is never 0, so the domain is $\mathbb{R}$.

34. Clearly $x\neq 0$. But we can also see that $x\neq -1$. Therefore, the domain is $(-\infty,-1)\cup(-1,0)\cup(0,\infty)$.

35. Division by 0 is not allowed, $x\neq 5$. The domain is $(-\infty,5)\cup(5,\infty)$.

36. This is $1/\sqrt{4-x^2}$ so the domain is all x such that $4-x^2 > 0$. Equivalently, $4 > x^2$ or $2 > |x|$. The domain is $(-2, 2)$.

37. To avoid division by 0, the domain is all $x \neq -2$. That is, $(-\infty, -2) \cup (2, \infty)$.

38. It must be the case that $x^2 + x - 2 = (x+2)(x-1) \geq 0$. Therefore, either $x \geq 1$, or $x \leq -2$. The domain is $(-\infty, -2] \cup [1, \infty)$.

39. First of all, $x \neq \pm 3$, because $x = \pm 3$ makes the denominator 0. In addition, it must be the case that $25 - x^2 \geq 0$. That is, $25 \geq x^2$ or $5 \geq |x|$, implying that $-5 \leq x \leq 5$. The domain is $[-5, -3) \cup (-3, 3) \cup (3, 5]$.

40. For the numerator, $9 - x^2 \geq 0$. That is, $9 \geq x^2$ or $3 \geq |x|$, and $-3 \leq x \leq 3$. For the denominator, $x^2 - 1 \geq 0$ or $x^2 \geq 1$, and $|x| \geq 1$. Therefore, the domain is $[-3, -1] \cup [1, 3]$.

41. The first 5 terms are 1/2, 2/3, 3/4, 4/5, and 5/6.

42. The first 5 terms: 1/2, 5/4, 7/8, 17/16, 31/32.

43. These are 4, 8, 14, 24, 42.

44. Simplify to $(2n+1)/2n-1)$: 3, 5/3, 7/5, 9/7, 11/9.

45. The first one is $1 + 2 + 4 = 7$. Then 15, 31, 63, and 127.

46. The first one is $1 \cdot 2 \cdot 3 = 6$. Then 24, 120, 720, and 5040.

47. Here are the first 5: 1, 2, 4, 8, $f_5 = 16$.

48. The first 5: 1, 3, 6, 10, $f_5 = 15$.

49. The first 5: 0, 2, 7, 18, $f_5 = 41$.

50. The first one is 1, then 2, 7/3, 12/5, and $f_5 = 41/17$.

51. $f^2(\sqrt{x}\,) = (2\sqrt{x} + 3)^2 = 4x + 12\sqrt{x} + 9$

52. $(f^2 - 4g)(x) = f^2(x) - 4g(x) = (2x+3)^2 - 4(1+x^2) = 12x + 5$

53. $(h \circ g)(x) = h(g(x)) = (1 - (1+x^2))/(1 + (1+x^2)) = -x^2/(2+x^2)$

54. $(g \circ f)(x) = g(f(x)) = 1 + (2x+3)^2 = 4x^2 + 12x + 10$

55. $(f \circ g^2)(x) = f(g^2(x)) = 2(1+x^2)^2 + 3 = 2x^4 + 4x^2 + 5$

56. $$\begin{aligned}(h \circ (2f))(x) = h(2f(x)) &= (1 - (2 \cdot (2x+3))/(1 + 2 \cdot (2x+3)) \\ &= (-5 - 4x)/(7 + 4x)\end{aligned}$$

57. $(f \circ g^{1/2})(\sqrt{x}\,) = f\big(\sqrt{g(\sqrt{x}\,)}\,\big) = 2\sqrt{1+x} + 3$

58. $(h \circ h)(x) = h(h(x)) = \dfrac{1 - \frac{1-x}{1+x}}{1 + \frac{1-x}{1+x}} = \dfrac{1 + x - (1-x)}{1 + x + (1-x)} = \dfrac{2x}{2} = x$

59. $\left(f^{-1} \circ \frac{1}{f}\right)(x) = f^{-1}(1/f(x)) = (1/f(x) - 3)/2 = (1/(2x+3) - 3)/2$

$$= -(3x+4)/(2x+3)$$

60. $(f \circ f \circ f)(x) = f(f(f(x))) = f(2(2x+3)+3) = 2(4x+9)+3 = 8x+21$

61. As s increases from 1 to 4, $f(s)$ decreases from $1/2$ to $1/5$, so $T = [1/5, 1/2]$. Since $t = 1/(1+f^{-1}(t))$, $1+f^{-1}(t) = 1/t$ and $f^{-1}(t) = 1/t - 1$.

62. As s increases from 1, $f(s)$ increases from 2, so $T = [2, \infty)$. Since $t = 3(f^{-1}(t))^2 - 1$, $(f^{-1}(t))^2 = (t+1)/3$ and, because $f^{-1}(t)$ is positive, $f^{-1}(t) = \sqrt{(t+1)/3}$.

63. As s increases from 0 to 1, $f(s)$ increases from 2 to 5, so $T = [2, 5]$. Since $t = s^2 + 2s + 2$, $s^2 + 2s + 2 - t = 0$ and $s = (-2 \pm \sqrt{4 - 4(2-t)})/2$. That is, $s = -1 \pm \sqrt{t-1}$. Because s is not negative, $s = -1 + \sqrt{t-1}$ and $f^{-1}(t) = -1 + \sqrt{t-1}$.

64. As s increases from 0 to 1, $f(s)$ decreases from 2 to $2/3$, so $T = (2/3, 2)$. Since $t = (1+s)/(2+s)$, $t(2+s) = 1+s$ and $s(t-1) = 1-2t$. That is, $s = (1-2t)/(t-1)$. Therefore, $f^{-1}(t) = -(2t-1)/(t-1)$.

65. As s increases from -1 towards ∞, $f(s)$ increases to 1, then decreases steadily towards 0. Therefore, f is onto, but it is not one-to-one. Observe, for example, that $f(-1) = f(1) = 1/2$.

66. As s increases from 1 to 100, $f(s)$ increases steadily from $f(1) = 1$ to $f(100) = 100 + \log_{10}(100) = 102$. Therefore, f is one-to-one, but it is not onto.

67. As s increases from 0 towards ∞, $f(s)$ increases from 0 towards 1, but it never gets there. Therefore, f is one-to-one, but it is not onto.

68. f is neither one-to-one nor onto. For example, $f(0) = f(2) = 2$ and $f(s)$ does not attain the value 0. (Note that $f(s) = (s-1)^2 + 1$, implying that $f(s) \geq 1$ for all s.)

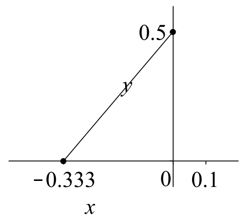
69. To get the graph of g from the graph of f, translate it 1 unit down and 2 units left.

70. To get the graph of g from the graph of f, translate it 2 units right.

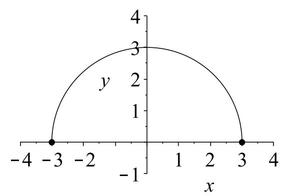
71. Since $g(x) = f(x-3) - 9$, to get the graph of g from the graph of f, translate it 9 units down and 3 units right.

72. To get the graph of g from the graph of f, translate it 3 units up and 3 units left.

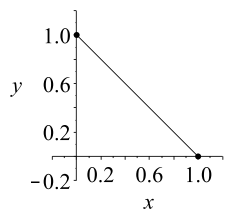
73. This is a portion of a line. It extends from $(-1/3, 0)$ to $(0, 1/2)$. See the picture.



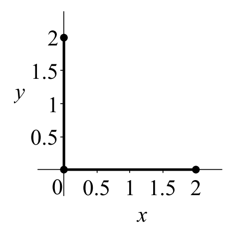
74. This is half of a circle of radius 3. It is traced out in the counterclockwise direction. See the picture.



75. This is a portion of a line. It extends from $(1, 0)$ to $(0, 1)$. See the picture.



76. This is a broken line segment. It extends from $(0, 2)$ to $(0, 0)$ to (2.0). See the picture.



77. The table below contains the values of the six trigonometric functions at the angle A. Note that $a = 3$, $b = 2$, and $c = \sqrt{13}$.

$\sin(A)$	$\cos(A)$	$\tan(A)$	$\cot(A)$	$\csc(A)$	$\sec(A)$
$3/\sqrt{13}$	$2/\sqrt{13}$	$3/2$	$2/3$	$\sqrt{13}/3$	$\sqrt{13}/2$

78. The table below contains the values of the six trigonometric functions at the angle A. Note that $a = 12$, $b = 5$, and $c = 13$.

$\sin(A)$	$\cos(A)$	$\tan(A)$	$\cot(A)$	$\csc(A)$	$\sec(A)$
$12/13$	$5/13$	$12/5$	$5/12$	$13/12$	$13/5$

79. The table below contains the values of the six trigonometric functions at the angle A. Note that $a = 1$, $b = 3$, and $c = \sqrt{10}$.

$\sin(A)$	$\cos(A)$	$\tan(A)$	$\cot(A)$	$\csc(A)$	$\sec(A)$
$1/\sqrt{10}$	$3/\sqrt{10}$	$1/3$	3	$\sqrt{10}$	$\sqrt{10}/3$

80. The table below contains the values of the six trigonometric functions at the angle A. Note that $a = 4$, $b = 3$, and $c = 5$.

$\sin(A)$	$\cos(A)$	$\tan(A)$	$\cot(A)$	$\csc(A)$	$\sec(A)$
$4/5$	$3/5$	$4/3$	$3/4$	$5/4$	$5/3$

81. The table below contains the values of the six trigonometric functions at the angle $\beta = \alpha + \pi/2$, were α is the radian measure of the angle A of Exercise 78. Note that $a = 12$, $b = 5$, $c = 13$, and adding $\pi/2$ does the following to the sine and cosine: $\sin(\alpha + \pi/2) = \cos(\alpha)$ and $\cos(\alpha + \pi/2) = -\sin(\alpha)$. The other values are obtained from these two.

$\sin(\beta)$	$\cos(\beta)$	$\tan(\beta)$	$\cot(\beta)$	$\csc(\beta)$	$\sec(\beta)$
$5/13$	$-12/13$	$-5/12$	$-12/5$	$13/5$	$-13/12$

82. The table below contains the values of the six trigonometric functions at the angle $\beta = \alpha + \pi$, were α is the radian measure of the angle A of Exercise 80. Note that $a = 4$, $b = 3$, $c = 5$, and adding π does the following to the sine and cosine: $\sin(\alpha + \pi) = -\sin(\alpha)$ and $\cos(\alpha + \pi) = -\cos(\alpha)$. The other values are obtained from these two.

$\sin(\beta)$	$\cos(\beta)$	$\tan(\beta)$	$\cot(\beta)$	$\csc(\beta)$	$\sec(\beta)$
$-4/5$	$-3/5$	$4/3$	$3/4$	$-5/4$	$-5/3$

83. $\tan(\pi/4) - \cot(3\pi/4) = 1 - (-1) = 2$

84. $\sin(\pi/6) + \cos(5\pi/3) = 1/2 + 1/2 = 1$

85. $\csc(\pi/6)\sec(3\pi/4) = 2 \cdot (-\sqrt{2}\,) = -2\sqrt{2}$

86. $\cot(\pi/3)\cot(4\pi/3) = (1/\sqrt{3}\,) \cdot (1/\sqrt{3}\,) = 1/3$

87. $r = \sin(2\alpha) = 2\sin(\alpha)\cos(\alpha) = 2 \cdot (3/4) \cdot \sqrt{1 - 9/16} = 3\sqrt{7}/8$

88. $$r = \cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha) = \cos^2(\alpha)(1 - \tan^2(\alpha)) = \frac{1 - \tan^2(\alpha)}{\sec^2(\alpha)}$$
$$= \frac{1 - \tan^2(\alpha)}{1 + \tan^2(\alpha)} = \frac{1 - 1/4}{1 + 1/4} = 3/5$$

89. $$r = \tan(2\alpha) = \frac{\sin(2\alpha)}{\cos(2\alpha)} = \frac{2\sin(\alpha)\cos(\alpha)}{\cos^2(\alpha) - \sin^2(\alpha)} = 2 \cdot \frac{\sqrt{1 - \cos^2(\alpha)} \cdot \cos(\alpha)}{2\cos^2(\alpha) - 1}$$
$$= 2 \cdot \frac{\sqrt{1 - 9/25} \cdot (3/5)}{2 \cdot (9/25) - 1} = 2 \cdot \frac{12/25}{-7/25} = -24/7$$

90. $$r = \cos(2\alpha + \pi/2) = -\sin(2\alpha) = -2\sin(\alpha)\cos(\alpha)$$
$$= -2\sqrt{1 - \cos^2(\alpha)} \cdot \cos(\alpha) = -2\sqrt{1 - 4/9} \cdot (2/3) = -4\sqrt{5}/9$$