Chapter 1 Thinking Critically

COOPERATIVE INVESTIGATION
The Gold Coin Game

1. Starting with seven markers, the first player should take three, leaving four on the desktop. Since the second player can only take one, two, or three markers, the first player can take the remaining marker or markers and so win the game.

2. Starting with 11 markers, the first player should take three, leaving eight on the desktop. Then the second player must leave five, six, or seven markers on the desktop and the first player can then leave just four as in step 1. Continuing as in step 1, the first player will win the game.

3. We see a pattern beginning to form and guess that, with 15 markers, the first player should again start by taking three markers. The second player must then leave 9, 10, or 11 markers. In each case, the first player can then reduce the number to eight. The play then continues as in step 2 and the first player again wins the game.

4. Yes, the strategy will work for 51 markers, and the first player will win. A winning position is any position where the number of markers is a multiple of 4 (4, 8, 12, 16, 20, 24, …). Presented with such a position and the possibility of taking only one, two, or three markers, a player can't help but create a position that can be immediately turned back into a winning position by the other player. Of course, if the game starts with the number of markers a multiple of four, then the second player can force a win, rather than the first.

Variation

Of course, the details of a similar game where the player who takes the last marker *loses* the game can vary widely. However, the most likely variation is to change just this one part of the rules still requiring the players to play alternately and to take only one, two, or three markers each time.

1. Suppose the game starts with six, seven or eight markers on the desktop. The first player should play so as to leave five markers. Then the second player must leave four, three, or two, and the first player can then leave just one marker, which the second player must take. Thus, the first player wins.

2. Suppose the game starts with 10, 11, or 12 markers on the desktop. This time the first player should play so as to leave nine. Then the second player must leave eight, seven, or six and, in each case, the first player can then leave just five and proceed to win the game as in step 1.

3. In this game, a winning position is any position where the number of markers on the desktop is one more than a multiple of 4 (1, 5, 9, 13, 17, 21, …). Faced with such a position and the possibility of taking only one, two, or three markers on any one move, a player must necessarily create a new non-winning position that can be immediately turned back into a winning position by the other player. Of course, if the starting number of markers on the desktop is one more than a multiple of four, then the second player can force a win, rather than the first.

Section 1.1
An Introduction to Problem Solving

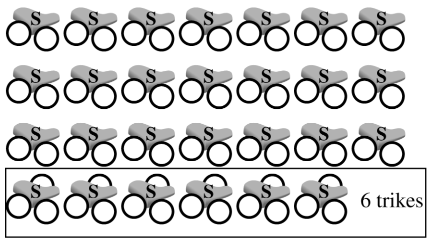
Problem Set 1.1

1. (a) Using guess and check:
Guess 14 bikes, 13 trikes. The number of wheels is $14 \times 2 + 13 \times 3 = 67$.
Too many wheels, too many trikes.
Guess again: 17 bikes, 10 trikes. The number of wheels is $17 \times 2 + 10 \times 3 = 64$.
Still too many wheels.
Guess again: 21 bikes, 6 trikes. The number of wheels is: $21 \times 2 + 6 \times 3 = 60$.
O.K.

(b)

Bikes	Trikes	Bike Wheels	Trike Wheels	Total Wheels
17	10	34	30	64
18	9	36	27	63
19	8	38	24	62
20	7	40	21	61
21	6	42	18	60

(c) Place two wheels next to each seat and then add a third wheel to as many seats as necessary to make 60 wheels.



(d) Yes. If all trikes have one wheel off the ground, there are 54 wheels touching the ground. To make 60 wheels there must be 6 wheels in the air—6 trikes.

2. This problem can be solved by Make a Table or Guess and Check. Using either method, the result is 62 students and 83 adults.
$62 \cdot \$3 + 83 \cdot \$5 = \$186 + \$415 = \$601$

3. **(a)** Suppose all 32 stamps were 18-cent stamps. The total worth of the stamps would be \$5.76. \$8.07 – \$5.76 = \$2.31. Since 29-cent stamps cost 11 cents more than 18-cent stamps, the total number of 29-cent stamps is 231 ÷ 11 = 21. Mr. Akika has 11 18-cent stamps and 21 29-cent stamps. (Note that this is Jennifer's strategy.)

(b) Answers will vary. One method is given in (a). Another possibility is to make an educated guess and then adjust the guess upward or downward depending on how the total value of the stamps under your guess compares to \$8.07.

4. **(a)** Guess at various numbers and ultimately arrive at $x = 13$.
$3 \cdot 13 - 11 = 39 - 11 = 28$

(b) Make a Table which starts with a number and then go up (if the first guess is low) or down (if the first guess is high.) Continue until you find $x = 13$.

Trial Number	Result
10	$3 \cdot 10 - 11 = 30 - 11 = 19$
11	$3 \cdot 11 - 11 = 33 - 11 = 22$
12	$3 \cdot 12 - 11 = 36 - 11 = 25$
13	$3 \cdot 13 - 11 = 39 - 11 = 28$

(c) Using algebra, if x is Toni's guess, then $3x - 11 = 28$, hence $3x = 39$ or $x = 13$.

5. Use the Make a Table strategy.

Dimes	Nickels	Pennies	Total Value
4	0	5	45¢
4	1	4	49¢

4 dimes are too many; try 3 dimes.

Dimes	Nickels	Pennies	Total Value
3	1	5	40¢
3	2	4	44¢
3	3	3	48¢

Xin has 3 dimes, 3 nickels, and 3 pennies.

6. Answers will vary. One possibility is given. 145 tickets were sold for the Spring concert at Willowbrook School. Tickets were priced at \$3 for students and \$5 for adults. The income from the sale of the tickets was \$601. How many student and adult tickets were sold?

7. Guess and Check or Make a Table.

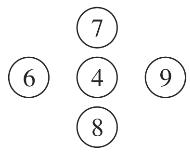
Guess	Multiply by 5	Subtract 8 to obtain result
10	50	42
11	55	47
12	60	52 ✓

8. Guess and Check or Make a Table.

Guess	Multiply by −2	Add 12 to obtain result
6	−12	0
7	−14	−2
8	−16	−4 ✓

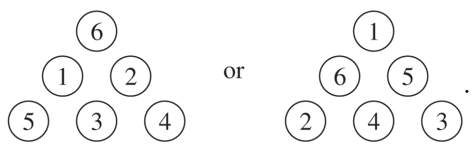
9. Answers will vary. Two possibilities are given. Who am I? If you multiply me by 4 and subtract 8, the result is 60. Answer: 17
Sally has 15 nickels and dimes worth a total of \$1.10. How many of each coin does she have? Answer: 8 nickels and 7 dimes.

10. **(a)** The only possible sums of 3 digits totaling 19 are:
$4 + 7 + 8 = 19$
$4 + 6 + 9 = 19$.
Since 4 is used twice, it must be in the middle.



(b) Since the two sums above are the only ways to obtain 19, there are no solutions other than interchanging the 6 and 9 or the 7 and 8 or rotating the figure. But these produce essentially the same solution.

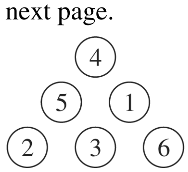
11. **(a)** Answers will vary. Two possibilities are



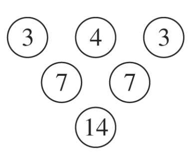
Another possibility is given in part (c).

(b) Yes.

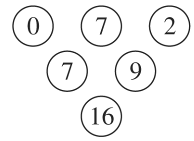
(c) Answers will vary. One method is given. First, choose a number which is to be the sum of the numbers on each side—say, 11. (Several choices are possible.) We need to write 11 as a sum of three numbers (1 through 6) in three different ways. 11 can be written as $6 + 4 + 1$, $6 + 3 + 2$, or $5 + 4 + 2$. Since 2, 4, and 6 each appear in two of the sums, we place 2, 4, and 6 in the "corner" circles and complete the diagram as shown on the next page.



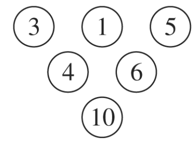
12. **(a)** Note that $7 - 4 = 3$, $3 + 4 = 7$, and $7 + 7 = 14$.



(b) Note that $9 - 7 = 2$, $16 - 9 = 7$, and $7 - 7 = 0$.



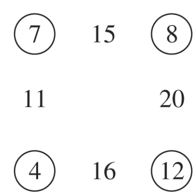
(c) This is one of many possible solutions.



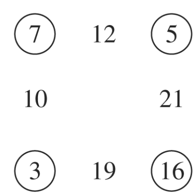
(d) The solutions given for parts (a) and (b) are the only solutions. There are multiple solutions to part (c).

13. Parts (a), (b), and (c) have more than one solution. You can place an arbitrary number in the upper left circle and then complete the rest of the circles.

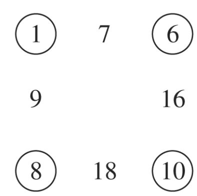
(a)



(b)

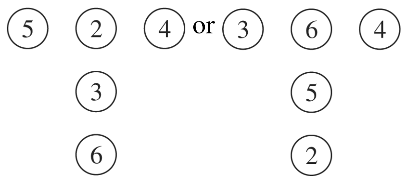


(c)

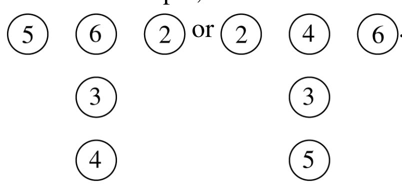


(d) Note that in each of (a), (b), and (c), the sum of the top and bottom numbers given is the sum of the left and right numbers; i.e., $11 + 20 = 15 + 16$, $10 + 21 = 12 + 19$, and $9 + 16 = 7 + 18$. For such a problem to have a solution, this must always be the case. Thus, there is no solution.

14. **(a)** Answers will vary. The key lies in finding two pairs of numbers that have the same sum—for example, 5 + 4 and 3 + 6 are equal. By placing the remaining number, 2, in the top middle circle, the diagram can be completed in several ways. Two alternatives are shown below.

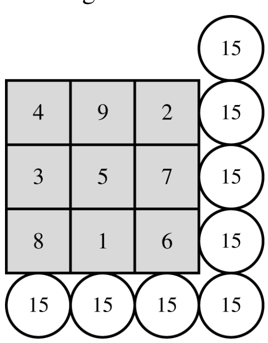


(b) Yes. For example,

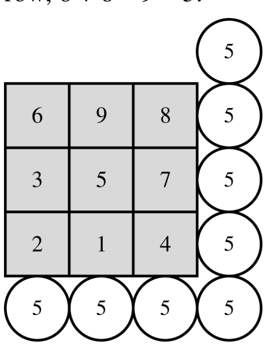


(c) No. The remaining numbers, 2, 4, 5, 6, must be separated into pairs that have the same sum. But, this is impossible with 2, 4, 5, and 6 since one sum will be even and one will be odd.

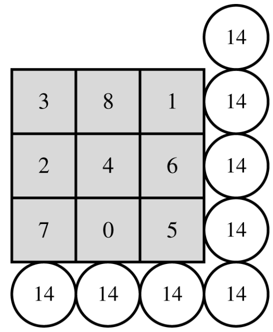
15. **(a)** Each sum is 15. In any magic square, the sum of the numbers in each row, column, and diagonal is the same.



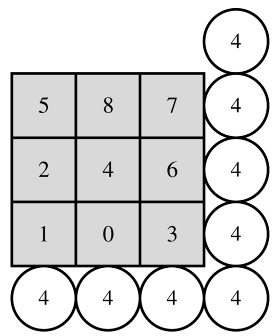
(b) Each result is 5. For example, in the top row, 6 + 8 – 9 = 5.



16. **(a)** Since the sum of the digits 0, 1, 2, 3, 4, 5, 6, 7, and 8 is 36 and the sum in each row must be the same, you can conclude that each row must have a sum equal to 12. One way to create a magic square is to subtract 1 from each entry in Problem 14(a).



(b) Using the process from Problem 14(b), interchange the 1 and 7 and the 3 and 5 obtaining:



Each result is 4, the sum of two end entries minus the middle entry.

17. **(a)** Since 2 + 3 = 5, 3 + 5 = 8, 5 + 8 = 13, and 8 + 13 = 21, the sequence is 1, 2, 3, 5, 8, 13, 21.

(b) Since 8 – 2 = 6, 6 + 8 = 14, 8 + 14 = 22, 14 + 22 = 36, and 22 + 36 = 58, the sequence is 2, 6, 8, 14, 22, 36, 58.

(c) This can be solved using the guess and check strategy. A more formal solution is as follows: Suppose the second number is n. Then the third number is $3 + n$, and the fourth number is 13. Therefore, $n + (3 + n) = 13$, so $n = 5$. The sequence is 3, 5, 8, 13, 21, 34, 55.

(d) Suppose the second number is n. Then the sequence is 2, n, $2 + n$, $2 + 2n$, $4 + 3n$, 26. Therefore, $(2 + 2n) + (4 + 3n) = 26$ or $6 + 5n = 26$ or $5n = 20$ or $n = 4$. The sequence is 2, 4, 6, 10, 16, 26.

(e) Suppose the second number is n. Then the sequence is 2, n, $2 + n$, $2 + 2n$, $4 + 3n$, 11. Therefore $(2 + 2n) + (4 + 3n) = 11$, so $6 + 5n = 11$ and $n = 1$. The sequence is 2, 1, 3, 4, 7, 11.

Section 1.2
Pólya's Problem-Solving Principles and the Common Core State Standards for Mathematical Practice

Problem Set 1.2

1. **(a)** No, because $5 \times 10 + 13 = 63$, not 48.

(b) We could use Guess and Check, Make a Table, or algebra to find Nancy's number.

(c) Using algebra, if x is Nancy's number, then $5x + 13 = 48$ gives $5x = 35$ or $x = 7$.

2. Using the Guess and Check and Make a Table strategies, we construct the following table.

Guess for Lisa's number	Number that results
1	$7 \cdot 1 - 4 = 3$
2	$7 \cdot 2 - 4 = 10$
3	$7 \cdot 3 - 4 = 17$

Lisa's number is 3.

3. Construct the following table:

Guess	Twice the guess plus 1	Three times the guess minus 5
4	9	7
5	11	10
6	13	13

Vicky's number must be 6.

4. **(a)** Yes. Yes. Juan's rule could have been either one of these. The result obtained by the first rule is $5n - 3$. The result obtained by the second rule is $5(n - 1) + 2 = 5n - 5 + 2 = 5n - 3$, which is the same as the first rule.
Without algebra, you can reason that multiplying one less than a number by five gives a result that is 5 less than the original number multiplied by 5. Then adding 2 gives a number which is 3 less than the original number multiplied by 5.

(b) Each result is one more than the square of the chosen number. In algebraic symbols, this is given by $n^2 + 1$.

(c) Starting with 0 and choosing the numbers in order is a good idea because it enables us to see patterns more clearly. The fact that Peter's numbers increase by 3 each time suggests that "multiply the number by 3" is part of the rule. The rule is: Multiply the chosen number by 3 and then add 7. In algebraic symbols, this is given by $3n + 7$.

5. The four-coin possibilities are:

QQQQ	$1.00	QDNN	0.45
QQQD	0.85	QNNN	0.40
QQQN	0.80	DDDD	0.40
QQDD	0.70	DDDN	0.35
QQDN	0.65	DDNN	0.30
QQNN	0.60	DNNN	0.25
QDDD	0.55	NNNN	0.20
QDDN	0.50		

Since $0.40 appears twice, there are 14 different amounts of money.

6. There are 49 ways, listed below in order of number of pennies used. For example, Q + D + 15P means a quarter, a dime, and 15 pennies.

2Q	9N + 5P	2D + 2N + 20P
Q + 2D + N	Q + D + N + 10P	D + 4N + 20P
Q + D + 3N	Q + 3N + 10P	6N + 20P
Q + 5N	4D + 10P	Q + 25P
5D	3D + 2N + 10P	2D + N + 25P
4D + 2N	2D + 4N + 10P	D + 3N + 25P
3D + 4N	D + 6N + 10P	5N + 25P
2D + 6N	8N + 10P	2D + 30P
D + 8N	Q + D + 15P	D + 2N + 30P
10N	Q + 2N + 15P	4N + 30P
Q + 2D + 5P	3D + N + 15P	D + N + 35P
Q + D + 2N + 5P	2D + 3N + 15P	3N + 35P
Q + 4N + 5P	D + 5N + 15P	D + 40P
4D + N + 5P	7N + 15P	2N + 40P
3D + 3N + 5P	Q + N + 20P	N + 45P
2D + 5N + 5P	3D + 20P	50P
D + 7N + 5P		

There are 49 different ways to make change.

7. 258, 285, 528, 582, 825, 852

8. 305, 350, 503, 530 (Note: A 3-digit number cannot start with 0)

9. Use the Make a Table strategy.

Number of dimes	Number of nickels	Number of pennies
2	0	1
1	2	1
1	1	6
1	0	11
0	4	1
0	3	6
0	2	11
0	1	16
0	0	21

10. Use the Make a Table strategy. Assume that the pearls and the bags are identical, so that, for example, 21, 1, 3 is considered the same as 21, 3, 1. Then we need list only the possibilities in which the number of pearls in bag 1 is at least as great as the number of pearls in bag 2, and the number of pearls in bag 2 is at least as great as the number of pearls in bag 3. We have the following:

Bag 1	Bag 2	Bag 3
23	1	1
21	3	1
19	5	1
19	3	3
17	7	1
17	3	5
15	9	1
15	7	3
15	5	5
13	11	1
13	9	3
13	7	5
11	11	3
11	9	5
11	7	7
9	9	7

11. **(a)** Look for two whole numbers whose product is 120.

1, 120 2, 60 3, 40 4, 30
5, 24 6, 20 8, 15 10, 12

(b) The perimeter is twice the sum of the length and width. Therefore, the $10 \text{ cm} \times 12 \text{ cm}$ rectangle has the smallest perimeter, 44 cm.

12. Make a Table.

Number pair	Sum
4, 24	28
6, 16	22
8, 12	20

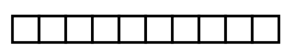
13. Find common multiples of 20 and 30 and determine the amount for which Peter worked 5 more days.

Common multiples of 20 and 30	Days worked Jill	Days worked Peter
60	2	3
120	4	6
180	6	9
240	8	12
300	10	15

Jill worked 10 days, and Peter worked 15 days.

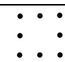
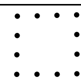
14. 10 days. (He reaches a *maximum* height of 3 feet on the first day, 4 feet on the second day, and so on.)

15. A diagram of the situation will show that Bob has to make 9 cuts to get 10 2-foot sections. Since each cut takes one minute, it will take Bob 9 minutes to do this.



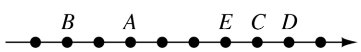
16. The number of posts is always one more than the number of sections. For example, one 20-foot section requires 2 posts and two sections require 3 posts. Since ten sections of fence are needed for the 200-foot fence, 11 posts will be required.

17. Try solving a simpler problem and make a diagram.

Number of sections on each side	Number of posts	Diagram
1	4	
2	8	
3	12	

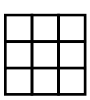
For 10 sections on each side, the number of posts is 40.

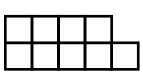
18. Though it may seem unusual, the simplest approach to this problem is that of drawing a diagram as in Example 1.5. Make the analogy between the race in Example 1.5 and the political race in the present problem. Draw a line with equally spaced points with the distance between consecutive points representing 1000 votes. Then place *A* (for Albright), *B*, *C*, *D*, and *E* on the line according to the statement of the problem.

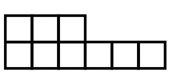


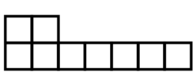
Place *A* at an arbitrary point on the line and *B* two units to *A*'s left since Albright finished 2000 votes ahead of Badgett, and so on. The completed diagram is as shown and the order of finishing from first to last is Dawkins, Chalmers, Ertl, Albright, and Badgett.

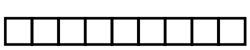
19. **(a)** The possible perimeters are 12, 14, 16, 18, and 20. See the diagrams. Note that the perimeter cannot be odd because the edges are adjoined in pairs. Of course, other diagrams are possible but none yields a perimeter other than those shown below.


perimeter = 12


perimeter = 14


perimeter = 16

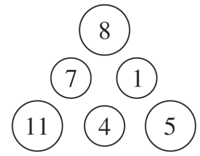

perimeter = 18


perimeter = 20

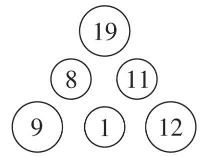
(b) The 3-by-3 square has the least perimeter.

20. Answers will vary significantly.

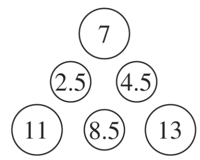
21. **(a)** Add: 7 + 1 = 8, 1 + 4 = 5, 4 + 7 = 11.



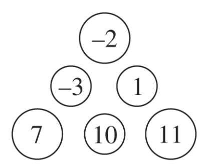
(b) Note that 11 + 1 = 12, 19 − 11 = 8, and 8 + 1 = 9.



(c) The sum of the three new numbers must be $\frac{7+11+13}{2} = 15.5$. Note that 15.5 − 7 = 8.5, 15.5 − 11 = 4.5, and 15.5 − 13 = 2.5.



(d) The sum of the three new numbers must be $\frac{-2+7+11}{2} = 8$. Note that 8 − (−2) = 10, 8 − 7 = 1, and 8 − 11 = −3.



22. Answers will vary significantly.

23. **(a)** The combinations are (skirt, red tee), (skirt, blue tee), (skirt, lime green tee), (shorts, red tee), (shorts, blue tee), (shorts, lime green tee). There are 2 × 3 = 6 combinations.

(b) There are 2 × 3 × 2 = 12 combinations.

24. There are several ways to determine the overall winner, and not all give the same answer. One possibility is to see who won the most events. According to this criterion, Sarah is the overall winner. However, since all events are to be weighed equally, suppose we make a table placing 1, 2, 3, or 4 in each column according as the corresponding individual finished 1^{st}, 2^{nd}, 3^{rd}, or 4^{th} in that event. Adding these numbers across each row gives numbers which take *all* events into account and the person with the least total score will have finished highest in the overall competition. In this case, Angel is the overall winner.

	50-m Dash	**Sit-Ups**	**600-m Run**	**Pull-Ups**	**Total Score**
Sarah	4	1	1	3	9
Jan	1	3	4	4	12
Angel	2	2	2	2	8
Mike	3	4	3	1	11

25. Use the Make a Table Strategy.

White Chairs	**Black Chairs**
1	0
1	4
5	4
5	8
9	8
9	12
13	12
13	16
17	16
17	20

The answer is B. There are 20 black chairs.

26. Examine the factors of 24 and 18 to find the greatest common factor of both numbers. The factors of 24 are 1, 2, 3, 4, 6, 8, and 24. The factors of 18 are 1, 2, 3, 6, 9, and 18. Therefore, the greatest common factor is 6. Judith Ann has three groups of 6, so Sandy must also have three groups. Sandy has three groups of 8. The answer is B.

Section 1.3
More Problem-Solving Strategies

Problem Set 1.3

1. **(a)** 2, 5, 8, 11, <u>14, 17, 20</u>.
Each succeeding term is 3 more than the preceding term.

(b) –5, –3, –1, 1, <u>3, 5, 7</u>.
Each succeeding term is 2 more than the preceding term.

(c) 1, 1, 3, 3, 6, 6, 10, <u>10, 15, 15</u>.
Notice that the third, fifth, and seventh terms in the sequence are determined by adding 2, 3, and 4 respectively to the second, fourth, and sixth terms.

The even numbered terms are the same as the preceding term, so the sixth term is 10. The seventh term is $10 + 5 = 15$, and the eighth term is also 15.

2. **(a)** 1, 3, 4, 7, 11, <u>18, 29, 47</u>.
Each term is the sum of the two immediately preceding terms. This is the beginning of a Fibonacci sequence.

(b) 2, 8, 32, 54, 128, <u>512, 2048, 8192</u>.
Each term is four times the preceding term.

(c) 1, 9, 25, 49, <u>81, 121, 169</u>.
The sequence consists of consecutive odd squares.

3. **(a)** Each term is 2 more than its predecessor. Since $35 = 5 + 30$ and $30 \div 2 = 15$, we conclude that 15 2s must be added to 5 to get 35. Thus, there are 15 terms after the first one, or 16 terms in all.

(b) Each term is 5 more than its predecessor. Since $46 = -4 + 50$ and $50 \div 5 = 10$, we conclude that 10 5s must be added to get –4 to get 46. There are 11 terms.

(c) Each term is 4 more than its predecessor. Since $67 = 3 + 64$ and $64 \div 4 = 16$, we conclude that 16 4s must be added to 3 to get 67. There are 17 terms.

4. **(a)** Yes, the sequence is arithmetic because the difference of consecutive terms is the constant –3.

(b) The difference between the first and last terms is $-52 - 8 = -60$. Thus, the number of terms is one more than $-60 \div (-3) = 20$, or 21 terms.

5. (a) The middle term on the left side of the *n*th equation is *n*. The next two lines are:
$1 + 2 + 3 + 4 + 5 + 4 + 3 + 2 + 1 = 25$
$1 + 2 + 3 + 4 + 5 + 6 + 5 + 4 + 3 + 2 + 1 = 36.$

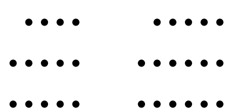
(b) Using the pattern observed, the sum is $100^2 = 10{,}000$.

(c) The sum is n^2.

6. (a) $1 + 3 + 5 + 7 + 5 + 3 + 1 = 9 + 16$
$1 + 3 + 5 + 7 + 9 + 7 + 5 + 3 + 1 = 16 + 25$

(b) $(n-1)^2 + n^2$

7. (a) Each pattern has one more column of three dots than the previous one. The next two patterns are shown.



(b) 2, 5, 8, 11, 14, 17

(c) The pattern is that each term is increased by 3. Thus, the sequence of numbers can be expressed as:
2, 2 + 3(1), 2 + 3(2), 2 + 3(3), ...
The 10th term is 2 + 3(9) = 29.
The 100th term is 2 + 3(99) = 299.

(d) Since 33 3s must be added to 2 to get 101, the number 101 must be the 34th term.

8. (a) 16 seems most likely.

(b)

n	1	2	3	4
2^n	2	4	8	16
$n^2 - n + 2$	2	4	8	14
$n^3 - 5n^2 + 10n - 4$	2	4	8	20

(c) The test writer implies that only one answer is possible. In fact, there are infinitely many rules for forming sequences that give 2, 4, and 8 as the first three terms followed by any number you might choose as the fourth entry! It would be much better if the test writer would write, "What is the most likely next term in the sequence 2, 4, 8, ...?

9. Notice the pattern. Except for the last column of the table, when you move right one square, the units digit increases by 1, and when you move down one square, the tens digit increases by 1. Hence, we fill in the arrays and determine the desired last number as shown.

(a)

53	54	55	56
			66
			76
			86

The desired number is 86.

(b)

34				
	45			
		56		
			67	
				78

The desired number is 78.

(c)

				37
			46	
		55		
	64			
73				

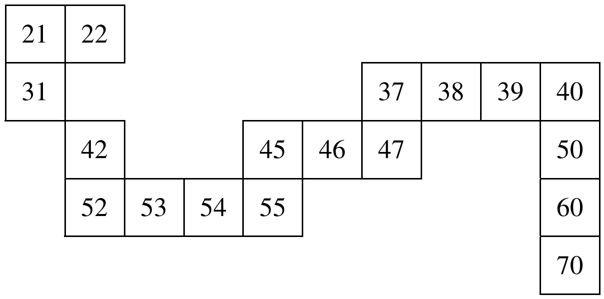
The desired number is 73.

10. (a)

42	43			
	53			
		64		
			75	
				86
			95	

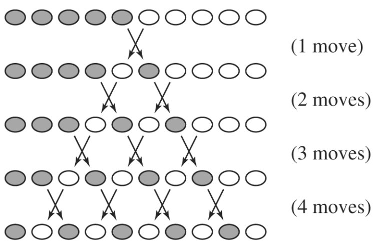
The desired number is 42.

(b)



The desired number is 70.

11. The least number of moves is 10. There are many ways to accomplish this in ten moves; one way is shown.



Note that 1 + 2 + 3 + 4 = 10.

12. **(a)** The left side of the equation has an additional square number added or subtracted in each step. Additions and subtractions alternate.
1 – 4 + 9 – 16 + 25 = 15
1 – 4 + 9 – 16 + 25 – 36 = –21

(b) 1 – 4 + 9 – 16 + 25 – 36 + 49 = 28
1 – 4 + 9 – 16 + 25 – 36 + 49 – 64 = –36

(c) n even: $1-4+9-\cdots-n^2=-\dfrac{n(n+1)}{2}$

n odd: $1-4+9-\cdots+n^2=\dfrac{n(n+1)}{2}$

13. **(a)** In the 1 by 3 figure, there is one 1 by 3 rectangle, two 1 by 2 rectangles, and three 1 by 1 rectangles. The number of rectangles is 1 + 2 + 3 = 6.
In the 1 by 4 figure, there is one 1 by 4 rectangle, two 1 by 3 rectangles, three 1 by 2 rectangles, and four 1 by 1 rectangles. The number of rectangles is 1 + 2 + 3 + 4 = 10.

(b) The figure measures 1 by 16, so the number of rectangles is

$1+2+3+\cdots+16=t_{16}=\dfrac{16\cdot 17}{2}=136$.

(c) The number of rectangles is $1+2+3+\cdots+n$. This is the nth triangular number, $t_n=\dfrac{n(n+1)}{2}$.
(See section 1.4 in the text.)

(d) Answers will vary. For example, a 1 by n strip has one 1 by n rectangle starting in square number one; two 1 by n – 1 rectangles starting in squares number 1 and 2; three 1 by n – 2 rectangles starting in squares number 1, 2, and 3; ...; and n 1 by 1 rectangles starting in squares number 1, 2, ..., n. Thus, the total number of rectangles is

$1+2+3+\cdots+n=\dfrac{n(n+1)}{2}$.

14. The number of paths from A to B, C, D, and E are 4, 2, 14, and 8 respectively.

15. There are 35 paths from A to B

16. There are 110 paths from A to B.

17. The possible paths are
1. $C\to A\to B\to E$
2. $C\to B\to E$
3. $C\to F\to D\to E$
4. $C\to F\to E$
5. $C\to F\to B\to E$

There are 5 paths from C to E.

18. **(a)** As shown in the diagram in the text, there are 6 chords.

(b) There are 45 chords.

(c) Each dot is connected to 99 other dots, so 99 chords end at each dot. By multiplying 100×99, we count each chord twice (since a chord has 2 endpoints), so the number of chords is $\dfrac{100\times 99}{2}=4950$.

(d) There are $\frac{n(n-1)}{2}$ chords.

(e) One way to explain this is given above in part (c). An alternate explanation follows. The number of chords determined by n dots is equal to the number of chords determined by $n - 1$ dots plus $(n - 1)$ more made by connecting the nth dot with each of the previous $n - 1$. Since 2 dots determine 1 chord and 3 dots determine $1 + 2 = 3$ chords, then n dots determine $1+2+3+\cdots+(n-1)$ $= \frac{(n-1)n}{2}$ chords.

19. **(a)** $\frac{10 \cdot 9}{2} = 45$ games

(b) $\frac{11 \cdot 10}{2} = 55$ games

(c) Yes. Each team can be represented as a dot on the circle and each segment represents a game played between the two teams.

20. **(a)** $2 + 3 + 3 + 4 = 12$
$5 + 6 + 6 + 7 = 24$
$11 + 12 + 12 + 13 = 48$
$15 + 16 + 16 + 17 = 64$
Sum $= 4 \times$ (upper right entry)

(b) $4 + 5 + 6 + 5 + 6 + 7 + 6 + 7 + 8 = 54$
$10 + 11 + 12 + 11 + 12 + 13 + 12 + 13 + 14 = 108$
$14 + 15 + 16 + 15 + 16 + 17 + 16 + 17 + 18 = 144$

(c) Sum $= 9 \times$ (upper right entry)

(d) $\text{Sum} = \left(\begin{array}{c}\text{number of}\\ \text{little squares}\end{array}\right) \times \left(\begin{array}{c}\text{upper}\\ \text{right entry}\end{array}\right)$

21. **(a)** Answers will vary widely.

(b) Answers will vary widely.

22. **(a)** Answers will vary widely.

(b) Answers will vary widely.

23. None. It's a hole!

24. There are always 12 in a dozen, regardless of price.

25. The coins are a penny and a quarter. The problem states that *one* of the coins is not a quarter. It does not say that neither coin was a quarter.

26. Each number in the sequence is 9 greater than its predecessor. Also, the numbers are the consecutive multiples of 9 from 9 to 90.

27. One possibility is simply to try each of the given rules to see which one generates Duane's sequence. Checking this way, we see that rule C generates the given pattern.
$4 \times 0 + 2 = 2$, $4 \times 2 + 2 = 10$,
$4 \times 10 + 2 = 42$, and $4 \times 42 + 2 = 170$.

Section 1.4
Algebra as a Problem-Solving Strategy

Problem Set 1.4

1. **(a)** Each pattern has one more column of dots than the previous one. The next three patterns are shown:

● ● ● ●   ● ● ● ● ●   ● ● ● ● ● ●
● ● ● ●   ● ● ● ● ●   ● ● ● ● ● ●

(b) 2, 4, 6, 8, 10, 12

(c) Each term is twice the number of the term. Therefore, the 10th term is $2(10) = 20$ and the 100th term is $2(100) = 200$.

(d) The nth even number.

(e) Divide 2402 by 2 and obtain 1201.

2. Let n = Toni's number. Toni doubles it to get $2n$, and then adds 11, ending with $2n + 11$. Thus, $2n + 11 = 39$ or $2n = 28$, so $n = 14$.

3. **(a)** Let m be Jackson's number. To triple m and subtract 13 means to compute $3m - 13$. Thus, $3m - 13 = 2$ or $3m = 15$, so $m = 5$ is Jackson's number.

(b) There are no solutions since the solution of the equation $3n - 13 = 4$ is $n = 17/3$, which is not an integer.

4. **(a)** Let x be Maria's integer. Maria finds that $x/2+12=10$ or $x + 24 = 20$, so $x = -4$ is Maria's integer.

(b) There are no integers that solve $2n + 12 = 15$ so there are no solutions. The answer of $n = -\frac{3}{2}$ is not a whole number. Another way of responding is to note that the left hand side of the equation has to be even whereas the other side is odd.

5. Each additional train requires 4 more matchsticks, so the formula has the form $t = k + 4c$ for some constant k. Evaluating this expression at $c = 1$, then $k + 4 = 5$, since the first pattern has $t = 5$ matchsticks. Therefore $k = 1$, and the desired formula is $t = 1 + 4c$.

6. (a) Each additional table increases the number of seats by 4. Thus, the number seated at n tables has the form $a + 4n$, for some a. Since 6 are seated at the first table (when $n = 1$), we see that $a = 2$. Thus, $2 + 4n$ people can be seated at n tables.

(b) To seat 24 people, n must be the smallest integer for which $2 + 4n$ is at least 24. This occurs at $n = 6$, and leave two empty places.

7. (a) Person A shakes with B and C. Since B and C have already shaken hands with A, only one shake remains, B with C. The total is 2 + 1 = 3.

(b) If the people are A, B, C, D, E, and F, then A shakes hands with the other five, and B has only four people left with whom to shake hands (C, D, E, and F). Similarly, C has only three handshakes, D has 2, E has 1, and there is nobody new left for F. Total is $5 + 4 + 3 + 2 + 1 = 15$.

(c) The logic is the same as in part (b). The first person shakes hands with the other 199, the second with 198, the third with 197, etc. until we get to the penultimate person who has one new hand to shake. The total is

$$199 + 198 + 197 + 196 + \ldots + 2 + 1 = \frac{(200)(199)}{2} = 19,990,$$

where we have used Gauss's Insight.

(d) The logic is the same as in part (c) The first person shakes hands with the other $n - 1$, the second with $n - 2$, the third with $n - 3$, etc., until we get to the penultimate (second to last) person who had only one new hand to shake. Thus, the total is

$$(n-1) + (n-2) + (n-3) + \ldots + 2 + 1 = \frac{n(n-1)}{2},$$

using Gauss's Insight.

8. There are really two groups of fifty who will do the handshake problem, but no one in either group will shake the hands of someone in the other group. We can think of this as two rooms of 50 people each, so find the number of handshakes in each room using the result in problem 7(d), then add the two answers. Since $n = 50$ for one of the groups, the answer for that group is $\frac{50(49)}{2}$. Thus, total number of handshakes in the room of 100 is

$$\frac{50(49)}{2} + \frac{50(49)}{2} = 2450.$$

9. There are 100 people in the room. If all of them shake hands exactly once, then (using the result in problem 7), there is a total of $100(99)/2 = 4950$ handshakes. However, since no husband and wife shake each other's hand, there are 50 less according to the condition of the problem. The answer is $4950 - 50 = 4900$.

10. Answers will vary. One possibility is given: 99 people consisting of 33 groups of triplets from different families are in a room. If no triplet shakes hands with his own family member, and everyone who can shake each other's hand does, how many handshakes are there?

11. Let c and g denote the number of chickens and goats, respectively. Then $c + g = 100$ and $2c + 4g = 286$, since there are 100 chickens and goats, and chickens have 2 feet each whereas goats have 4 feet each. From the first equation, $c = 100 - g$, which can be substituted into the second equation to obtain $2(100 - g) + 4g = 286$. This is equivalent to $200 - 2g + 4g = 286$, which simplifies to $2g = 286 - 200 = 86$. Therefore, there are $g = \frac{86}{2} = 43$ goats and $c = 100 - g = 100 - 43 = 57$ chickens.

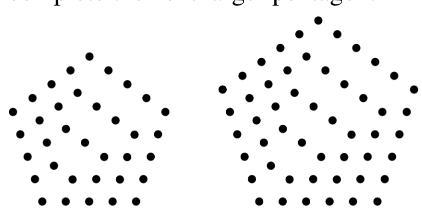
12. (a) $21 + 23 + 25 + 27 + 29 = 125$

(b) n^3

(c) $1^2 - 1 + 1 = 1$, $2^2 - 2 + 1 = 3$,
$3^2 - 3 + 1 = 7$, $4^2 - 4 + 1 = 13$ and
$1^2 + 1 - 1 = 1$, $2^2 + 2 - 1 = 5$,
$3^2 + 3 - 1 = 11$, $4^2 + 4 - 1 = 19$

(d) $\dfrac{n[(n^2 - n + 1) + (n^2 + n - 1)]}{2} = n(n^2) = n^3$

13. (a) In each figure, dots are added to the upper left, upper right, and lower right sides to complete the next larger pentagon.



(b) 1, 5, 12, 22, 35, 51, ...

(c) 1 + 4 + 7 + 10 + 13 = 35
1 + 4 + 7 + 10 + 13 + 16 = 51

(d) 10th term = 1 + 3(9) = 28

(e) Use Gauss's Insight:

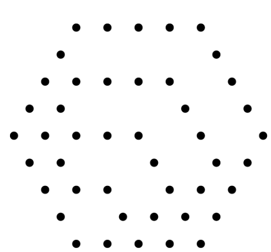
$$\begin{array}{l} s = 1 + 4 + 7 + \cdots + 28 \\ \underline{s = 28 + 25 + 22 + \cdots + 1} \\ 2s = 29 + 29 + 29 + \cdots + 29 \end{array}$$

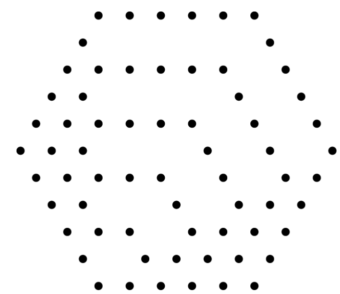
$$\text{Sum} = \frac{(10)(29)}{2} = 145$$

(f) nth term = 1 + 3(n – 1) = 3n – 2

(g) Using Gauss's Insight and the result from part (f), there are n terms of (3n – 1). The sum is $\dfrac{n(3n-1)}{2}$. Therefore,
$p_n = \dfrac{n(3n-1)}{2}$.

14. (a)





(b) 1, 6, 15, 28, 45

(c) $1 + 5 + 9 + \cdots + 37 = 190$

(d) The nth term is 1 + (4n – 3) = 4n – 2. Using Gauss's Insight to compute $h_n = 1 + 5 + 9 + \cdots + (4n - 3)$, we have n sums of 4n – 2, so
$h_n = \dfrac{n(4n-2)}{2} = n(2n-1)$.

15. Using the notation in Example 1.3, we have $a + b = 16, a + c = 11,$ and $b + c = 15.$ Subtracting the second equation from the first equation gives $b - c = 5$. Adding this result to the third equation yields $2b = 20$, so $b = 10$. Substituting this value into the first equation gives $a = 6$. Substituting $a = 6$ into the second equation gives $c = 5$.

16. (a) The entries in the second row are x + 8 and 9, so (x + 8) + 9 = 23, or equivalently, x + 17 = 23. Subtracting 17 from both sides gives x = 6. The entries in the second row are then 14 and 9, respectively.

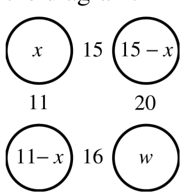
(b) The entries in the second row are 2 + y and y + 4. Therefore, (2 + y) + (y + 4) = 16, or equivalently 2y + 6 = 16. Therefore 2y = 10 and y = 5. The two entries in the middle row are 7 and 9, respectively.

(c) The middle entry in the bottom row is z – 4, so that the other entry in the second row is (z – 4) + 6; that is, z + 2. Then z + (z + 2) = 12, or equivalently, 2z + 2 = 12. Thus, z = 5 and the other entry in the second row is 7. The middle entry in the bottom row is 1.

17. Suppose that x denotes the value in the lower small circle. Then the entries in the other small circles are $17 - x$ and $26 - x$, giving the equation $(17 - x) + (26 - x) = 11$. This simplifies to $43 - 2x = 11$, or $2x = 43 - 11 = 32$. Therefore, $x = 16$, and the entries in the other two circles are $17 - 16 = 1$ and $26 - 16 = 10$. Alternatively, one can work clockwise to see that the upper left small circle is $17 - x$ and therefore the remaining small circle value is $11 - (17 - x) = x - 6$. Then $(x - 6) + x = 26$, or $2x - 6 = 26$. As before, $x = 16$.

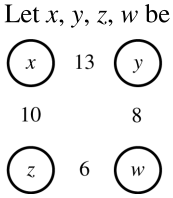
18. Let x be the unknown value in the lowermost circle. Working clockwise, the entries in the other small circles are $13 - x$, $13 + x$, $18 - x$, and $x - 1$. Thus, $x + (x - 1) = 23$, or equivalently, $2x = 24$ and $x = 12$. In clockwise order from the bottom, the entries are then 12, 1, 25, 6, and 11.

19. **(a)** Let x and w be integers which complete the diagram:



Looking at either $(11 - x) + w = 16$ or $(15 - x) + w = 20$ gives $w = 5 + x$. Thus x can be any integer and $w = x + 5$.

(b) Let x, y, z, w be integers in the circles:



The conditions of the problem are $x + y = 13,\ x + z = 10,\ y + w = 8,$ and $w + z = 6.$ Subtracting the second equation from the first gives $y - z = 3$. Subtracting the fourth equation from the third gives $y - z = 2$. It is impossible for $y - z = 3$ and $y - z = 2$. Thus, there are no solutions.

20. Let s be Dana's score on the fourth exam. Since her overall average must be ≥ 90, s must satisfy the inequality $\frac{78 + 86 + 94 + s}{4} \geq 90$, or $258 + s \geq 360$, so that $s \geq 102$. Unless Dana can write a perfect paper and get two extra credit points, an A is out of reach.

21. **(a)**

n	1	2	3	4	5	6
n^2	1	4	9	16	25	36
$(n+1)^2$	4	9	16	25	36	49
difference	3	5	7	9	11	13

(b) The difference is

$$(n+1)^2 - n^2 = (n^2 + 2n + 1) - n^2 = 2n + 1$$

22. Let x = the number of chickens and y = the number of pigs. There are 37 animals, so $x + y = 37$. Since there are two feet for each chicken (giving $2x$ total chicken feet) and four feet per pig (giving $4y$ total pig feet), $2x + 4y = 98$ or $x + 2y = 49$. Because the first equation is the same as $y = 37 - x$, substituting this into the last equation gives $x + 2(37 - x) = 49$ or $-x + 74 = 49,$ so $-x = -25$ or $x = 25.$ Thus, there are 25 chickens and 12 pigs. As a check, $25(2) + 12(4) = 50 + 48 = 98.$

23. **(a)** $23 + 32 = 55$
$42 + 24 = 66$
$17 + 71 = 88$
$51 + 15 = 66$
$67 + 76 = 143$

All of the answers are divisible by 11.

(b) The numbers given by the decimal description are $10a + b$ and $10b + a$. Thus, their sum is

$$\begin{aligned}(10a + b) + (10b + a) &= 11a + 11b \\ &= 11(a + b),\end{aligned}$$

which is divisible by 11.

24. **(a)** $32 - 23 = 9$
$42 - 24 = 18$
$71 - 17 = 54$
$51 - 15 = 36$
$76 - 67 = 9$

All of the answers are divisible by 9.

(b) The numbers given by the decimal description are $10a + b$ and $10b + a$. Thus, their difference is

$$\begin{aligned}(10a + b) - (10b + a) &= 9a - 9b \\ &= 9(a - b),\end{aligned}$$

which is divisible by 9.

25. Let L and W denote the length and width of the rectangle, respectively, and S the length of the sides of the square. Since the rectangle is 3 times as long as it is wide, $L = 3W$. Therefore, the perimeter of the rectangle is $2L + 2W = 6W + 2W = 8W$, and its area is $LW = 3W^2$. The perimeter of the square is $4S$, and its area is S^2. We know both the rectangle and the square have the same perimeter, so $8W = 4S$, or $2W = S$. Also, the area of the square is 4 square feet larger than the area of the rectangle, so $S^2 = 3W^2 + 4$. By substitution, $(2W)^2 = 3W^2 + 4$, or $4W^2 = 3W^2 + 4$. Therefore, $W^2 = 4$, and the positive width of the rectangle is $W = 2$. Its length is $L = 3W = 6$. The square has sides of length $S = 2W = 4$.

26. **(a)** $x^2 - y^2 = (x - y)(x + y) = 2(x + y)$

(b) If $x > y$ and x and y differ by n, then the difference in their squares is n times their sum. This is true by factoring as in part (a).

27. If x is the number of students and y is the number of adults, then $x + y = 145$ and $3x + 5y = 601$. Since $x + y = 145$, $y = 145 - x$. Using substitution, we have
$3x + 5(145 - x) = 601 \Rightarrow$
$3x + 725 - 5x = 601 \Rightarrow -2x = -124 \Rightarrow x = 62$
and $y = 145 - 62 = 83$.
There were 62 students and 83 adults.

28. $x = 2n$ so $x^2 = 4n^2$, thus x^2 is 4 times an integer, n^2.

29. The sum of the two equations gives twice the heart is 20. Thus, the heart is 10 and the star is 4.

30. Roberto has 80 cards left after giving his sister 10. He gave $80/5 = 16$ cards to each of his friends. (Note: This assumes that his sister is not one of the four friends.) This is choice B.

31. **(a)** Daniel is 10 feet away, and Christine is 19 feet away.

(b) Daniel has moved 10 + 5 + 2.5 + 1.25 = 18.75 feet and is 1.25 feet from the door. Christine has moved four feet and is 16 feet from the door. Daniel is closer.

(c) Daniel is correct. He will get as close to the door as he wishes, but will never actually get there.

32. If x is the number of rides on Tuesday, then $62 = 16 + 2x$. This give $46 = 2x$ and $23 = x$. The answer is choice D.

Section 1.5
Additional Problem-Solving Strategies

Problem Set 1.5

1. The second player can always add a sufficient number of tallies to reach a multiple of 5 at each step. This will force the first player to go over 30.

2. To get exactly 12, Annabelle must have exactly 8 to win when it is her turn. This forces her partner to then add one, two or three dots. To get to exactly 8, Annabelle goes to exactly 4. Thus when the first player (Andy) puts down one, two, three dots, Annabelle can add enough so she has 4 dots on the table. She can always win under these rules.

3. **(a)** Work backwards. Start with 39 and perform the inverse operations of those indicated.
$39 \times 2 = 78$
$78 + 18 = 96$
$96 \div 6 = 16$
$16 - 7 = 9$
The input number is 9.

(b) Start with 48.
$48 \times 2 = 96$
$96 + 18 = 114$
$114 \div 6 = 19$
$19 - 7 = 12$
The input number is 12.

(c) Answers will vary. The guess and check method is one possibility.

(d) Let x be the input. After two stages, we have $6(x + 7)$. The output is
$$(6(x+7) - 18) \div 2 = \frac{6x + 42 - 18}{2} = 3x + 12.$$
If the output is 39 as in (a), then $3x + 12 = 39$, so $3x = 27$ or $x = 9$. If the output is 48 as in (b), then the input satisfies $3x + 12 = 48$, so $3x = 36$ or $x = 12$.

4. The easiest way of doing the problem is using algebra. If the input is x and output is y, then the machine starts with x and each step follows as shown below.

$$x \to x-7 \to 6(x-7) \to \frac{6(x-7)}{3} \to$$
$$\frac{6(x-7)}{3}+14 = 2(x-7)+14 = 2x = y$$

(a) $y = 39$ means that $x = 19.5$.

(b) $y = 48$ means that $x = 24$.

(c) Answers will vary. Guess and Check or Working Backwards are two other possibilities.

(d) The Use a Variable method is given at the beginning of the solution.

5. (a) $$\frac{6(3^2)-18}{2} = \frac{6(9)-18}{2} = \frac{54-18}{2} = \frac{36}{2} = 18$$

(b) If the input is x and output is y, then the machine starts with x and each step follows as shown below.

$$x \to x^2 \to 6x^2 \to 6x^2-18 \to$$
$$\frac{6x^2-18}{2} \to 3x^2-9 = y$$

6. (a) $$\frac{[6(3)]^2-18}{2} = \frac{18^2-18}{2} = \frac{324-18}{2} = \frac{306}{2} = 153$$

(b) If the input is x and output is y, then the machine starts with x and each step follows as shown below.

$$x \to 6x \to (6x)^2 \to 36x^2-18 \to$$
$$\frac{36x^2-18}{2} \to 18x^2-9 = y$$

(c) No, reversing the first two stages will usually result in two different machines.

7. (a) Work backward. Before the last jump, Josh had \$16, since $16 \times 2 = 32$. Before the second jump, Josh had $\frac{1}{2}(16+32) = \$24$. Before the first jump, Josh had $\frac{1}{2}(24+32) = \$28$. He started with \$28.

(b) Work backward two more steps. $\frac{1}{2}(28+32) = 30$ and $\frac{1}{2}(30+32) = 31$, so he had \$31.

8. (a) Working backward we have this sequence of results:
263, $2 \cdot 263 = 526$, $526 - 29 = 497$,
$497 \div 71 = 7$. Liping's number is 7.

(b) Let x = Liping's number. The processes that Liping does are $x \to 71x \to 71x+29 \to (71x+29) \div 2$. Thus,

$\frac{71x+29}{2} = 263$ is the equation for Liping's number. Therefore,
$71x+29 = 526$ or $71x = 497$ so $x = 7$.

9. Since the number is greater than 20, less than 35, and divisible by 5, it must be either 25 or 30. Since the sum of the digits is 7, it must be 25. Not all information was needed—for example, we did not use the first and second clues.

10. Since the number is a multiple of 11 between $4 \cdot 5 = 20$ and $7 \cdot 8 + 23 = 79$, it must be 22, 33, 44, 55, 66, or 77. Since it is a multiple of 3, it is 33 or 66. Since it is not even, it is 33. This is the only possibility.

11. Since the number is an even number between $8^2 = 64$ and $9^2 = 81$, it must be 64, 66, 68, 70, 72, 74, 76, 78, or 80. Since it is not divisible by 4, it must be 66, 70, 74, or 78. Since it is not divisible by 3, it is either 70 or 74. Either answer, 70 or 74, is possible.

12. Coats: Since Joe was wearing Moe's coat, Hiram must have been wearing Joe's coat. Therefore, Moe was wearing Hiram's coat. Hats: Since Joe was wearing Hiram's hat, Moe must have been wearing Joe's hat. Therefore, Hiram was wearing Moe's hat. To summarize, Moe was wearing Hiram's coat and Joes' hat. Hiram was wearing Joe's coat and Moe's hat.

13. If the cards start out in the order a, b, c, d, e, f, g, h, i, j then they end up in the order i, g, e, c, a, b, d, f, h, j. These need to correspond to 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Therefore, $a = 4$, $b = 5$, $c = 3$, and so on. The cards must start in the order 4, 5, 3, 6, 2, 7, 1, 8, 0, 9. The work backward strategy is also effective on this problem.

14.

	Choc. Malt	Straw. Shake	Banana Split	Walnut Cone
Aaron	X	X	X	O
Boyd	X	X	O	X
Carol	O	X	X	X
Donna	X	O	X	X

Aaron had the walnut cone. Boyd had the banana split. Carol had the chocolate malt. Donna had the strawberry shake.

15. Since Anne's husband is an only child and Will is Josie's brother, Taneisha is not married to Will. Combining this fact with the other clues, we immediately have:

	Kitty	Sarah	Josie	Taneisha
David	X	X	O	X
Will	X	O	X	X
Floyd	X	X	X	O
Gus	O	X	X	X

Completing the chart, we see that the following couples are married: Josie and David, Sarah and Will, Taneisha and Floyd, Kitty and Gus. (Note: The fact that Taneisha has two brothers is irrelevant.)

16. **(a)** Let n be the desired number. The successive questions and answers narrow the possibilities for n as follows:

$1 \le n \le 512$
$1 \le n \le 256$
$128 < n \le 256$
$192 < n \le 256$
$192 < n \le 224$
$192 < n \le 208$
$200 < n \le 208$
$200 < n \le 204$
$200 < n \le 202$

Since $n \ne 202$, then $n = 201$.

(b) 13 questions are required in each case. This is because $8192 = 2^{13}$, and each question is chosen to eliminate half of the possibilities. Since $2^{12} < 8000 < 2^{13}$, 13 questions are required in the 8000th case.

(c) 20 questions are sufficient to solve a problem with $2^{20} = 1{,}048{,}576$ possibilities. In this case, the number of possibilities that are eliminated is 1,048,575.

17. To number pages 1 through 9 takes $9 \cdot 1 = 9$ digits. To number pages 10 through 99 takes $90 \cdot 2 = 180$ digits. This leaves $867 - 180 - 9 = 678$ to number 3- digit pages. Thus, there are $678 \div 3 = 226$ 3-digit pages and $226 + 90 + 9 = 325$ pages in the book.

18. **(a)** 3. Use the Pigeonhole Principle—in this case, the children are the "pigeons" and genders are the "holes."

(b) 11. If there were only 5 boys and 5 girls, this would be 10 students—and the 11th student will always be sufficient to ensure that there are at least 6 of one gender.

19. **(a)** 366. Use the Pigeonhole Principle—in this case, the people are the "pigeons" and the 365 birthdays of the year are the "holes."

(b) 731. With 730 in a room, it is possible that there are exactly 2 people with each of the birthdays: January 1, January 2, ..., December 31.

20. Answers will vary.

21. If the difference $a - b$ is divisible by 10, that means that $a - b$ is a multiple of 10. In any set of 11 natural numbers, at least two of the numbers must have the same units digit which implies that their difference is a multiple of 10.

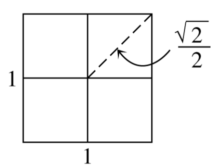
22. **(a)** Every natural number belongs to one of the six groups (or "pigeonholes") listed below:

Group	**Units Digit is:**
A	0
B	1 or 9
C	2 or 8
D	3 or 7
E	4 or 6
F	5

Since there are 7 numbers and only 6 groups, the pigeonhole principle implies that there must be at least two numbers which belong to the same group. If these two numbers have the same units digit, then their difference is divisible by 10. If their units digits are different, then their sum will have a units digit 0 and therefore be divisible by 10.

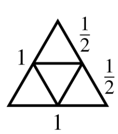
(b) Answers will vary. For example, 20, 21, 22, 23, 24, 25.

23.



If five points are chosen in a square with diagonal of length $\sqrt{2}$, then, by the Pigeonhole Principle, at least two of the points must be in or on the boundary of one of the four smaller squares shown. The farthest these two points can be from each other is $\sqrt{2}/2$ units, if they are on opposite corners of the small square.

24.



The first pigeonhole is the interior triangle including its boundary, and the three remaining pigeonholes are the outer triangles, each with two exterior boundaries. Two of the points must be in or on the boundary of one of the pigeonholes, and since each pigeonhole is an equilateral triangle with side length 1/2 meter, these two points can be no more than 1/2 meter apart.

25. If the cups of marbles are arranged as described, each cup will be part of three different groups of three adjacent cups. The sum of all marbles in all groups of three adjacent cups is $3 \cdot (10 \cdot 11/2) = 165$, since each cup of marbles is counted three times. With the marble count of 165 and 10 possible groups of three adjacent cups, by the Pigeonhole Principle, at least one group of three adjacent cups must have 17 or more marbles, since $165 \div 10 = 16.5 > 16$.

26. 23, because it is possible to have 2 boxes that contain 240 apples, 2 boxes that each contain 241 apples, ..., 2 boxes that each contain 250 apples, for a total of 22 boxes.

27. The number of people at the party with no friends is none, exactly one, or 2 or more.
Case (i): If everyone has at least one friend, then each of the 20 people at the party has between 1 and 19 friends, inclusive. By the Pigeonhole Principle, at least two of them have the same number of friends.
Case (ii): If exactly one person has no friends, then each of the other 19 people has 1 to 18 friends at the party. By the Pigeonhole Principle, at least two of them have the same number of friends.
Case (iii): If 2 or more people have no friends, then they have the same number of friends at the party.

28. Answers will vary. Sample answer: The number of hairs on any human's head is considerably less than the population of New York. Therefore, at least 2 people in New York have precisely the same number of hairs on their heads.

29. Working backward, we have the following sequence of numbers: 3, 6, 12, 24. Dan baked 24 cookies.

30.	tug-of-war	relay race	rope skipping
Emily	X	O	X
Mei	O	X	X
Andrew	X	X	O

Mei's favorite game is A, tug-of-war.

31. The answer is D.

Section 1.6
Reasoning Mathematically

Problem Set 1.6

1. **(a)**

$$1 \times 8 + 1 = 9$$
$$12 \times 8 + 2 = 98$$
$$123 \times 8 + 3 = 987$$

(b) It looks as if the value of each expression can be determined from the value of the previous expression. The digits for each value start at 9 and decrease by one. To find the new value, just insert another digit to the end of the previous value that is one less than the last digit in the previous value. Note that the number of digits in the value is the number that is added in the expression.

(c)

$$1234 \times 8 + 4 = 9876$$
$$12{,}345 \times 8 + 5 = 98{,}765$$
$$123{,}456 \times 8 + 6 = 987{,}654$$
$$1{,}234{,}567 \times 8 + 7 = 9{,}876{,}543$$
$$12{,}345{,}678 \times 8 + 8 = 98{,}765{,}432$$
$$123{,}456{,}789 \times 8 + 9 = 987{,}654{,}321$$

2. The same answer, 1089, is obtained for every choice of *abc*. Note that 99 should be written as 099.

3. **(a)**

$1 \times 1089 = 1089$	$6 \times 1089 = 6534$
$2 \times 1089 = 2178$	$7 \times 1089 = 7623$
$3 \times 1089 = 3267$	$8 \times 1089 = 8712$
$4 \times 1089 = 4356$	$9 \times 1089 = 9801$
$5 \times 1089 = 5445$	

(b) No. Patterns emerge. For example, the first two digits increase by 1 and the last two digits decrease by 1.

(c) The first and last products are reversals, so are the second and eighth, etc. The product 5445 is a palindrome—that is, it is its own reversal.

4. **(a)** The values at each given point are zero.

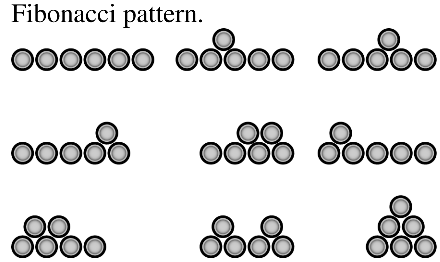
(b) No. Answers will vary. Sample answer: The value of $f(0)$ is -36.

5. The three points *P*, *Q*, and *R* are on a line.

6. No, the three-cut pie is a counterexample of the doubling conjecture. From the conjecture we would expect 8 pieces would be created by a three-cut pie; however either 6 or 7 pieces of pie are created.

7. **(a)** There are $F_5 = 5$ arrangements of five logs, supporting the generalization.

However, there are nine arrangements of six logs, instead of 8 as suggested by the Fibonacci pattern.

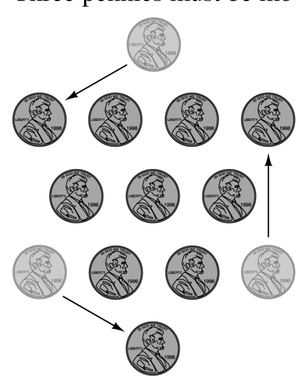


(b) The number of ways to stack *n* logs in two layers is given by the *n*th Fibonacci number F_n.

8. The number of ways to arrange a flagpole and guy wires on *n* blocks under the conditions given is given by the *n*th Fibonacci number F_n.

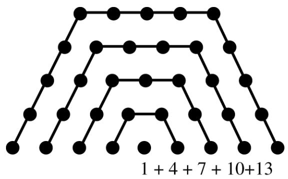
9. **(a)** Two pennies must be moved.

(b) Three pennies must be moved.

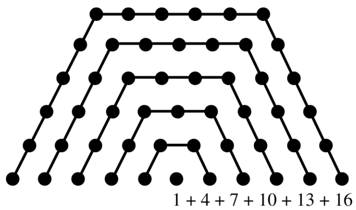


(c) Five pennies must be moved to invert a 15-penny triangle. In general, it can be shown that a triangle with n pennies requires $n/3$ pennies to be moved, where any remainder of the division is dropped. For example, $\frac{10}{3} = 3\frac{1}{3}$ so the triangle of 10 pennies requires 3 moves, and a 15 penny triangle requires $\frac{15}{3} = 5$ pennies to be moved.

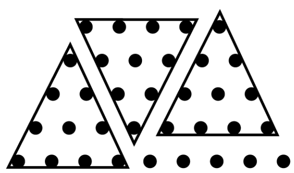
10. **(a)** The 1 is represented by the dot in the middle of the bottom row. The 4 is represented by the 4 adjacent dots, the 7 by the next layer of dots, and so on.



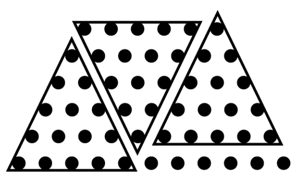
(b) The next layer has 16 dots, so $P_6 = P_5 + 16 = 35 + 16 = 51.$



(c)



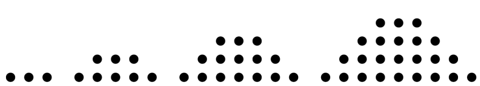
(d)



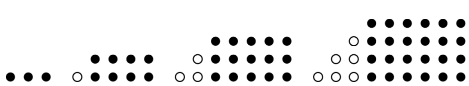
(e) The inductive pattern suggests that

$$P_n = n + 3t_{n-1} = n + 3\frac{(n-1)n}{2}$$
$$= \frac{3n^2}{2} - \frac{n}{2} = \frac{n(3n-1)}{2}$$

11. **(a)**



(b)



(c) The nth trapezoidal number is $n(n + 2)$ by inductive reasoning.

12. **(a)** The starting position and 15 moves to interchange the frogs are shown in a table:

○	○	○		●	●	●
○	○	○	●		●	●
○	○		●	○	●	●
○		○	●	○	●	●
○	●	○		○	●	●
○	●	○	●	○		●
○	●	○	●	○	●	
○	●	○	●		●	○
○	●		●	○	●	○
	●	○	●	○	●	○
●		○	●	○	●	○
●	●	○		○	●	○
●	●	○	●	○		○
●	●	○	●		○	○
●	●		●	○	○	○
●	●	●		○	○	○

(b) Eight moves are required.

(c) It requires $n(n + 2)$ moves to interchange n frogs of each color.

13. If n is a multiple of three, then $n = 3s$ for some whole number s and $n^2 = (3s)^2 = 9s^2$, which shows that n^2 is a multiple of 9.

14. Assume that n is even. Then $n = 2s$ for some whole number s. But then $n^2 = 4s^2 = 2(2s^2)$, which implies that n^2 is even. But this is false since we are given that n^2 is odd. Therefore, the assumption that n is even must be false. Thus, n is odd, as we were to prove.

15. Assume that n satisfies $2n + 16 = 35$. Since one side of the equation is $2(n + 8)$ is an even number (it is divisible by 2) and the other side, 35, is odd, there can be no solution to this equation. This is a proof by contradiction.

Chapter 1 Review Exercises

1. *Strategy 1*: Use trial and error and make a chart similar to the one below:

Number of 8′ boards	Number of 10′ boards	Total number of feet
45	45	810 (too small)
40	50	820 (too small)
35	55	830 (too small)
30	60	840 (too small)
28	62	844 (o.k.)

Therefore, there were 28 eight-foot boards.

Strategy 2: Use reasoning. If all boards were 8 feet long, the total length would be $90 \cdot 8 = 720$. He has $844 - 720 = 124$ "extra" feet. Since each ten-foot board has 2 "extra" feet, the number of ten-foot boards is $124 \div 2 = 62$. Therefore, there were 28 eight-foot boards.

2. **(a)** Answers will vary. Two solutions are

$$\begin{array}{r} 379 \\ 462 \\ +158 \\ \hline 999 \end{array} \qquad \begin{array}{r} 179 \\ 368 \\ +452 \\ \hline 999 \end{array}$$

(b) Yes. The digits in each column can be arranged in any order. (Other answers, such as 198 + 267 + 534, are also possible.)

(c) No. The hundreds column digit must sum to 8 to allow for a carry from the tens column. If the digit 1 is not in the hundreds column, the smallest that this sum can be is 2 + 3 + 4 = 9. Note that if 2, 3, and 4 are used, it will be impossible to construct a sum that does not require carrying from the tens digit. Thus, the digit 1 must be in the hundreds column.

3. There are 9 ways to produce 21¢ in change. They are listed in order of number of pennies used:

4N + P	3N + 6P	D + 11P
D + 2N + P	D + N + 6P	N + 16P
2D + P	2N + 11P	21P

4. $5 \times 4 \times 3 = 60$

5. The flower bed plus the walkway has a total area of $12 \times 14 = 168$ square feet, and the flower bed alone has an area of $8 \times 10 = 80$ square feet. The area of the walkway is $168 - 80 = 88$ square feet.

6. Karen's number is 9. Work backward. Add 7 to the result, 11, to obtain 18. Divide by 2 to obtain 9.

7. **(a)** Multiply by 5, then subtract 2.

(b) Answers will vary. A good strategy is to give Chanty consecutive whole numbers starting with 0.

8. **(a)** Multiply each term by 2 to get the next term: 3, 6, 12, <u>24</u>, <u>48</u>, <u>96</u>.

(b) Since $16 \div 4 = 2^2$, multiply each term by 2 to get the next term: 4, <u>8</u>, 16, <u>32</u>, <u>64</u>, <u>128</u>.

(c) Since $216 \div 1 = 6^3$, multiply each term by 6 to get the next term: 1, <u>6</u>, <u>36</u>, 216, <u>1296</u>, <u>7776</u>.

(d) Since $1250 \div 2 = 5^4$, multiply each term by 5 to get the next term: 2, <u>10</u>, <u>50</u>, <u>250</u>, 1250, <u>6250</u>.

(e) Since $7 \div 7 = 1^5$, multiply each term by 1 to get the next term: 7, <u>7</u>, <u>7</u>, <u>7</u>, <u>7</u>, 7.

9. We make a table to show all possibilities and use the clues to delete those that are impossible and, hence, those that are certain. The steps in the argument are numbered and the numbers in the table indicate the corresponding conclusions at each step.

	Doctor	Engineer	Teacher	Lawyer	Writer	Painter
Kimberly	no (5)	no (7)	no (1)	yes (8)	no (1)	yes (3)
Terry	yes (5)	yes (7)	no (6)	no (8)	no (2)	no (3)
Otis	no (4)	no (6)	yes (6)	no (6)	yes (2)	no (3)

(*continued on next page*)

(*continued*)

(1) By (b), Kimberly is neither the teacher nor the writer.

(2) By (e), Terry is not the writer. Therefore, Otis is the writer.

(3) By (f), neither Otis nor Terry is the painter. So, Kimberly is the painter.

(4) By (g), Otis is not the doctor.

(5) By (d), since the doctor hired the painter (Kimberly) and the doctor is not Otis, Terry is the doctor and Kimberly is not the doctor.

(6) Since Kimberly is not the teacher and, by (a), the doctor (Terry) had lunch with the teacher, Otis is the teacher and Terry is not. Also, since Otis has just two jobs, it follows that he is neither the engineer nor the lawyer.

(7) By (c), the painter (Kimberly) is related to the engineer. Therefore, Kimberly is not the engineer and so Terry is.

(8) Since Terry is the doctor and engineer he is not the lawyer. Thus, finally, Kimberly is the lawyer and the table now shows who holds what jobs.

10. **(a)** In the nth equation, we add the "next" n even numbers:

$$14 + 16 + 18 + 20 = 4^3 + 4$$
$$22 + 24 + 26 + 28 + 30 = 5^3 + 5$$
$$32 + 34 + 36 + 38 + 40 + 42 = 6^3 + 6$$

(b) $92 + 94 + 96 + 98 + 100 + 102 + 104 + 106 + 108 + 110 = 10^3 + 10$

11. Complete the chart below and then generalize from the results.

Number of chords	Number of regions	Number of intersections	Number of segments
0	1= 0 + 1	0	0
1	2 = 1 + 1	0	1
2	4 = 3 + 1	1	4
3	7 = 6 + 1	3	9
4	11 = 10 + 1	6	16
5	16 = 15 + 1	10	25
6	22 = 21 + 1	15	36
⋮	⋮	⋮	⋮
n	$\frac{n(n+1)}{2}+1$	$\frac{n(n-1)}{2}$	n^2

(a) $\frac{n(n+1)}{2}+1$

(b) $\frac{n(n-1)}{2}$

(c) Each chord is divided into n segments, for a total of n^2 small segments.

12. Let x be Bernie's weight. Bernie's weight is also represented by $90+\frac{x}{2}$. Then, by the condition of the problem, $x=90+\frac{x}{2}$, which yields $2x=180+x$ or $x=180$ pounds.

13. **(a)** A one-car train uses 6 toothpicks to form the hexagon. Adding a square + hexagon combination requires an additional 8 toothpicks, so the trains with 1, 3, 5, 7, … cars use 6, 6 + 8, 6 + 8 + 8, 6 + 8 + 8 + 8, … toothpicks. In general, a train with $2m + 1$ cars will require $6 + 8m$ toothpicks, where $m = 0, 1, 2, 3, \ldots$. A two-car train uses 9 toothpicks, so trains with 2, 4, 6, 8, … cars use 9, 9 + 8, 9 + 8 + 8, 9 + 8 + 8 + 8, … toothpicks. In general, a train with $2m + 2$ cars uses $9 + 8m$ toothpicks for $m = 0, 1, 2, 3, \ldots$.

(b) Since $9 + 8m$ is always an odd number, a train with 102 toothpicks has an odd number of cars, say $2m + 1$. Then $6 + 8m = 102$, or $8m = 96$. Therefore, $m = 12$, and there are $2(12) + 1 = 25$ cars in the train.

14. **(a)** Guessing will come up with 8 for one of the values and −1 for the other.

(b) Let x and y be the numbers. Then $x+y=7$ and $x-y=9$. Adding the equations gives $2x=16$, so $x=8$. Substituting this value into either equation gives $y=-1$. The solution checks.

15. **(a)** There are four "pigeonholes" (suits), so draw 5 cards.

(b) If only 8 cards are drawn, there could be 2 of each suit. Therefore, draw 9 cards.

(c) If one drew 48 cards, one might get everything except the aces. Therefore, to be absolutely sure of getting two aces, one must draw 50 cards.

16. 17, since it is possible that the first 16 books chosen are 4 each from the 4 types of books.

17. **(a)**

$$67 \times 67 = 4489$$
$$667 \times 667 = 444,889$$
$$6667 \times 6667 = 44,448,889$$

(b) $6,666,667 \times 6,666,667$
$= 44,444,448,888,889.$

The patterns observed suggest that:

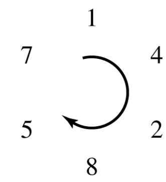
- the number of 4s is one more than the number of 6s in one of the factors.
- the number of 8s is the same as the number of 6s.
- the eights are followed by a single 9.

However, without knowing *why* the pattern holds, or doing the actual calculation, one cannot be completely sure that the guess is correct.

18. **(a)**

$$1 \times 142,857 = 142,857$$
$$2 \times 142,857 = 285,714$$
$$3 \times 142,857 = 428,571$$
$$4 \times 142,857 = 571,428$$
$$5 \times 142,857 = 714,285$$

(b) All of the above answers are obtained by starting at an appropriate place in the following circle.



The only remaining place to start is at 8 so we guess that
$6 \times 142,857 = 857,142$. This checks.

(c) The answer to $7 \times 142,857$ is not clear since there is no unused starting digit in the cycle. In fact, $7 \times 142,857 = 999,999$.

(d) Apparent patterns may be misleading; they may eventually break down.

19. If n is odd then $n = 2s + 1$ for some whole number s.

$$n^2 = (2s+1)^2 = 4s^2 + 4s + 1$$
$$= 4(s^2 + s) + 1 = 4\left[2 \cdot \frac{s(s+1)}{2}\right] + 1$$
$$= 8\left[\frac{s(s+1)}{2}\right] + 1$$

Since one of any two consecutive whole numbers must be even, $\frac{s(s+1)}{2}$ must be a whole number, say q. Thus $n^2 = 8q + 1$.

20. No, 12 is a multiple of 6 but the sum of its digits is 3, which is not divisible by 6.