CHAPTER 2

The Derivative

Section 2.1 Tangent Lines and Rates of Change

Suggested Time Allocation: 1 to $1\frac{1}{2}$ lectures

Teaching Plan : This section develops some close connections among three problems:

1. finding the tangent line to a curve;
2. determining the velocity of an object moving along a straight line; and
3. expressing the rate of change of one variable with respect to another.

Similar limits are involved in the solutions of these problems. (The problem of finding a tangent line to a curve was used to motivate the concept of a limit. If you covered that material, you can save time by recalling the discussion.)

Key points to emphasize are:

- The slope of a tangent line to a curve can be obtained as a limit of the slopes of secant lines to the curve.
- The slope of a secant line on a position versus time curve can be interpreted as an average velocity over a time interval, and the slope of a tangent line can be interpreted as an instantaneous velocity.
- The slope of a secant line on a curve $y = f(x)$ can be interpreted as an average rate of change of y with respect to x, and the slope of a tangent line can be interpreted as an instantaneous rate of change.
- The proper handling of units in applications of rates of change.

Margin Notes:

$$v_{\text{ave}} = \frac{\text{displacement}}{\text{time elapsed}} = \frac{f(t_0) - f(t_0+h)}{-h} = \frac{f(t_0+h) - f(t_0)}{h}$$

For part (a) write

$$r_{\text{ave}} = \frac{f(3+2) - f(3)}{2} = \frac{f(5) - f(3)}{2} = \frac{26 - 10}{2} = 8$$

and for part (b) write

$$\begin{aligned} r_{\text{inst}} &= \lim_{h\to 0} \frac{f(-4+h) - f(-4)}{h} = \lim_{h\to 0} \frac{[(-4+h)^2 + 1] - 17}{h} \\ &= \lim_{h\to 0} \frac{-8h + h^2}{h} = \lim_{h\to 0}(-8+h) = -8 \end{aligned}$$

Sample Assignment: Exercises 1, 3, 5, 7, 9, 11, 17, 25, 27, 29, 30

Section 2.2 The Derivative Function

Suggested Time Allocation: 1 to $1\frac{1}{2}$ lectures

Teaching Plan: This is the section in which the derivative function is defined formally. There is a lot of material so you may want to spend one lecture on the important concepts, and allot an additional half lecture to more computational examples. Comparing the behavior of the functions $\sin(1/x)$, $x\sin(1/x)$, and $x^2\sin(1/x)$ near the origin is a good way of distinguishing the concepts of continuity and differentiability.

Key points to emphasize are:

- Definition 2.2.1 and the role of the derivative as a "slope–producing function."
- Equation (3) for the tangent line.
- Example 1 or something comparable.
- The tangent line to the line $y = mx + b$ coincides with the line itself.
- The interpretation of the derivative function as the velocity function in the case of rectilinear motion (Equation (4)).
- The notion of differentiability and what it means geometrically, along with some of the ways in which a function can fail to be differentiable .
- The relationship between differentiability and continuity.
- Various derivative notations.
- Differentiability at the endpoints of a closed interval.

Sample Assignment: Exercises 1, 3, 5, 7, 9, 13, 15, 19, 22, 23, 31, 33, 35, 41, 45, 46

Section 2.3 Introduction to Techniques of Differentiation

Suggested Time Allocation: $\frac{1}{2}$ to 1 lecture

Teaching Plan: This section develops techniques of differentiation such as the power rule, the extended power rule, the constant multiple rule and the sum and difference rules. (These techniques enable one to differentiate any function that can be expressed as a linear combination of integral powers of x.) Consider presenting the proofs of Theorem 2.3.2 and Formula (9) of Theorem 2.3.5, leaving the proofs of the remainder of the theorems as a reading assignment. Students can also be asked to read the material that introduces the notation for higher order derivatives.

Key points to emphasize are:

- All of the stated theorems and some sample proofs.
- As you work through the examples, you might reinforce the theorems by reiterating their verbal forms (highlighted in gray).
- Notations for higher order derivatives.
- All of the examples or comparable examples.

Margin Notes:

Strictly speaking, since the expression 0^0 is undefined, we can interpret Formula (2) as a special case of (5) only if $x \neq 0$. In Section 3.6 we will view the expression 0^0 as an indeterminate form.

Sample Assignment: Exercises 1, 3, 5, 9, 11, 13, 15, 17, 23, 37, 39, 41, 43, 47, 49, 52, 53, 59, 61, 67, 69

Section 2.4 The Product and Quotient Rules

Suggested Time Allocation: $\frac{1}{2}$ to 1 lecture

Teaching Plan: This section introduces the product and quotient rules for differentiation. Consider skipping the proofs or presenting the proof of the product rule only. The focus should be on the effective application of the rules to examples.

Key points to emphasize are:

- The derivative of a product is in general *not* the product of the derivatives, and the derivative of a quotient is in general not the quotient of the derivatives.
- The verbal forms (highlighted in gray) of the product and quotient rules.
- All of the examples or comparable examples. (Illustrate the product rule for a function that is the product of three factors.)

Margin Notes:

Applying the quotient rule:

$$\left(\frac{1}{g}\right)' = \frac{g \cdot 0 - 1 \cdot g'}{g^2} = -\frac{g'}{g^2}$$

Therefore, $f'(x) = -\dfrac{2x}{(x^2+1)^2}$.

Sample Assignment: Exercises 1–25 (odd), 29–39 (odd), 43

Section 2.5 Derivatives of Trigonometric Functions

Suggested Time Allocation: $\frac{1}{2}$ lecture

Teaching Plan: This section contains the derivative formulas for the six trigonometric functions. It's worth the time to give the proof for the derivative of $\sin x$. Be sure to point out the importance of the now familiar limits

$$\lim_{h \to 0} \frac{\sin h}{h} = 1 \quad \text{and} \quad \lim_{h \to 0} \frac{1 - \cos h}{h} = 0$$

in the proof.

Key points to emphasize are:

- Derivative Formulas (3–8).
- Caution students about making "sign" errors when using the formulas.

Margin Notes:

If $x = \pi/4$ was substituted into $f'(x)$ before computing $f''(x)$ then we would incorrectly conclude that $f''(\pi/4) = 0$.

The top of the mass passes through its rest position when $s = -3\cos t = 0$ or equivalently when $\cos t = 0$. If $\cos t = 0$ then $\sin t = \pm 1$ and the speed has its maximum value of $|v| = |3\sin t| = 3$.

Sample Assignment: Exercises 1, 3, 5, 9, 11, 13, 15, 17, 19, 23, 25, 27, 29, 31, 33, 39

Section 2.6 The Chain Rule

Suggested Time Allocation: 1 lecture

Teaching Plan: This section is concerned with the chain rule, which many beginning students find difficult. It will be important that the student learn to apply the chain rule using dependent variables (as in Formulas (1) and (3)) and without using dependent variables). A nice application is to show that the quotient rule is a consequence of the product rule and the chain rule by writing $f(x)/g(x) = f(x) \cdot [g(x)]^{-1}$.

Key points to emphasize are:

- Motivate the chain rule by pointing out how rates of change multiply. (See the discussion that precedes Theorem 2.6.1.)
- The complete proof of Theorem 2.6.1 (Appendix D) is technical and can safely be omitted. However, you may want to motivate Theorem 2.6.1 with the following argument that works provided

  $$\frac{du}{dx} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \neq 0$$

  In this case, $\Delta u \neq 0$ for Δx close to 0 and we can write

  $$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \right)$$

  Since u is differentiable at x, u is continuous at x (Theorem 2.2.3) and thus $\Delta u \to 0$ as $\Delta x \to 0$. Therefore,

  $$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \right) = \left(\lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \right) \left(\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \right) = \frac{dy}{du} \cdot \frac{du}{dx}$$

- At some point, students have to be weaned from introducing dependent variables every time they want to apply the chain rule, so you may want to emphasize the verbal description of the chain rule.
- Generalized derivative formulas.

Margin Notes:

(Second margin note) If $y = f(g(x))$ and $u = g(x)$ then $y = f(u)$ and

$$\frac{d}{dx}[f(g(x))] = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = f'(u) \cdot \frac{du}{dx} = f'(g(x))g'(x)$$

Sample Assignment: Exercises 1–17 (odd), 21, 23, 25, 27, 32, 34, 35, 39, 41, 45, 53, 57, 59, 68, 69, 75

[Note: This is a long sample assignment because the exercise set is so rich in important ideas and techniques. If you feel that its length is a problem then you will have to eliminate some exercises or merge some of them with later assignments.]
