2

Signals and Systems

SIGNALS AND THEIR PROPERTIES

Solution 2.1

(a) $\delta_s(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m, y - n) = \sum_{m=-\infty}^{\infty} \delta(x - m) \cdot \sum_{n=-\infty}^{\infty} \delta(y - n)$, therefore it is a separable signal.

(b) $\delta_l(x, y)$ is separable if $\sin(2\theta) = 0$. In this case, either $\sin\theta = 0$ or $\cos\theta = 0$, $\delta_l(x, y)$ is a product of a constant function in one axis and a 1-D *delta* function in another. But in general, $\delta_l(x, y)$ is not separable.

(c) $e(x, y) = \exp[j2\pi(u_0 x + v_0 y)] = \exp(j2\pi u_0 x) \cdot \exp(j2\pi v_0 y) = e_{1D}(x; u_0) \cdot e_{1D}(y; v_0)$, where $e_{1D}(t; \omega) = \exp(j2\pi\omega t)$. Therefore, $e(x, y)$ is a separable signal.

(d) $s(x, y)$ is a separable signal when $u_0 v_0 = 0$. For example, if $u_0 = 0$, $s(x, y) = \sin(2\pi v_0 y)$ is the product of a constant signal in x and a 1-D sinusoidal signal in y. But in general, when both u_0 and v_0 are nonzero, $s(x, y)$ is not separable.

Solution 2.2

(a) Not periodic. $\delta(x, y)$ is non-zero only when $x = y = 0$.

(b) Periodic.

$$\text{comb}(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m, y - n)$$

For arbitrary integers M and N

$$\begin{aligned}
\text{comb}(x + M, y + N) &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m + M, y - n + N) \\
&= \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \delta(x - p, y - q) \text{ [let } p = m - M, q = n - N] \\
&= \text{comb}(x, y)
\end{aligned}$$

So the smallest period is 1 in both x and y directions.

(c) Periodic. Let $f(x+T_x, y) = f(x, y)$, we have

$$\sin(2\pi x)\cos(4\pi y) = \sin(2\pi(x+T_x))\cos(4\pi y).$$

Solve the above equation, we have $2\pi T_x = 2k\pi$ for arbitrary integer k. So the smallest period for x is $T_{x0} = 1$. Similarly we can find the smallest period for y is $T_{y0} = 1/2$.

(d) Periodic. Let $f(x+T_x, y) = f(x, y)$, we have

$$\sin(2\pi(x+y)) = \sin(2\pi(x+T_x+y)).$$

So the smallest period for x is $T_{x0} = 1$ and the smallest period for y is $T_{y0} = 1$.

(e) Not periodic. We can see this by contradiction. Suppose $f(x, y) = \sin(2\pi(x^2+y^2))$ is periodic, then there exist some T_x such that $f(x+T_x, y) = f(x, y)$:

$$\begin{aligned}\sin(2\pi(x^2+y^2)) &= \sin(2\pi((x+T_x)^2+y^2)) \\ &= \sin(2\pi(x^2+y^2+2xT_x+T_x^2))\end{aligned}$$

In order for the above equation to hold, we must have $2xT_x + T_x^2 = k$ for some integer k. The solution for T_x depends on x. So $f(x, y) = \sin(2\pi(x^2+y^2))$ is not periodic.

(f) Periodic. Let $f_d(m+M, n) = f_d(m, n)$, we have

$$\sin\left(\frac{\pi}{5}m\right)\cos\left(\frac{\pi}{5}n\right) = \sin\left(\frac{\pi}{5}(m+M)\right)\cos\left(\frac{\pi}{5}n\right)$$

Solve for M, we have $M = 10k$ for any integer k. The smallest period for both m and n is 10.

(g) Not periodic. Analog to part (f), by letting $f_d(m+M, n) = f_d(m, n)$, we have

$$\sin\left(\frac{1}{5}m\right)\cos\left(\frac{1}{5}n\right) = \sin\left(\frac{1}{5}(m+M)\right)\cos\left(\frac{1}{5}n\right)$$

The solution for M is $M = 10k\pi$. Since $f_d(m, n)$ is a discrete signal, its period must be an integer if it is periodic. There is no integer k that solves the equality for $M = 10k\pi$ for some M. So, $f_d(m, n) = \sin\left(\frac{1}{5}m\right)\cos\left(\frac{1}{5}n\right)$ is not periodic.

Solution 2.3

(a)

$$\begin{aligned}E_\infty(\delta_s) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta_s^2(x, y)\,dx\,dy \\ &= \lim_{X\to\infty}\lim_{Y\to\infty}\int_{-X}^{X}\int_{-Y}^{Y}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x-m, y-n)\,dx\,dy \\ &= \lim_{X\to\infty}\lim_{Y\to\infty}(2\lfloor X\rfloor+1)(2\lfloor Y\rfloor+1) \\ &= \infty\end{aligned}$$

where $\lfloor X\rfloor$ is the greatest integer that is smaller than or equal to X.

$$\begin{aligned}
P_\infty(\delta_s) &= \lim_{X\to\infty}\lim_{Y\to\infty}\frac{1}{4XY}\int_{-X}^{X}\int_{-Y}^{Y}\delta_s^2(x,y)\,dx\,dy \\
&= \lim_{X\to\infty}\lim_{Y\to\infty}\frac{1}{4XY}\int_{-X}^{X}\int_{-Y}^{Y}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x-m,y-n)\,dx\,dy \\
&= \lim_{X\to\infty}\lim_{Y\to\infty}\frac{(2\lfloor X\rfloor+1)(2\lfloor Y\rfloor+1)}{4XY} \\
&= \lim_{X\to\infty}\lim_{Y\to\infty}\left\{\frac{4\lfloor X\rfloor\lfloor Y\rfloor}{4XY}+\frac{2\lfloor X\rfloor+2\lfloor Y\rfloor}{4XY}+\frac{1}{4XY}\right\} \\
P_\infty(\delta_s) &= 1
\end{aligned}$$

(b)

$$\begin{aligned}
E_\infty(\delta_l) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\delta(x\cos\theta+y\sin\theta-l)|^2\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x\cos\theta+y\sin\theta-l)\,dx\,dy \\
&\overset{①}{=} \begin{cases}\int_{-\infty}^{\infty}\frac{1}{|\sin\theta|}\,dx, & \sin\theta\neq 0\\[2ex] \int_{-\infty}^{\infty}\frac{1}{|\cos\theta|}\,dy, & \cos\theta\neq 0\end{cases} \\
E_\infty(\delta_l) &= \infty
\end{aligned}$$

① comes from the scaling property of the point impulse. The 1-D version of Eq. (2.8) in the text is $\delta(ax)=\frac{1}{|a|}\delta(x)$.

Suppose $\cos\theta\neq 0$,

$$\delta(x\cos\theta+y\sin\theta-l)=\frac{1}{|\cos\theta|}\delta\left(x+y\frac{\sin\theta}{\cos\theta}-\frac{l}{\cos\theta}\right)$$

therefore,

$$\int_{-\infty}^{\infty}\delta(x\cos\theta+y\sin\theta-l)dx=\frac{1}{|\cos\theta|}.$$

$$\begin{aligned}
P_\infty(\delta_l) &= \lim_{X\to\infty}\lim_{Y\to\infty}\frac{1}{4XY}\int_{-X}^{X}\int_{-Y}^{Y}|\delta(x\cos\theta+y\sin\theta-l)|^2dx\,dy \\
&= \lim_{X\to\infty}\lim_{Y\to\infty}\frac{1}{4XY}\int_{-X}^{X}\int_{-Y}^{Y}\delta(x\cos\theta+y\sin\theta-l)dx\,dy
\end{aligned}$$

without loss of generality, assume $\theta=0$ and $l=0$, so we have $\sin\theta=0$ and $\cos\theta=1$. Therefore:

$$P_\infty(\delta_l) = \lim_{X\to\infty}\lim_{Y\to\infty}\frac{1}{4XY}\int_{-X}^{X}\int_{-Y}^{Y}\delta(x)\,dx\,dy$$

$$
\begin{aligned}
&= \lim_{X\to\infty}\lim_{Y\to\infty}\frac{1}{4XY}\int_{-Y}^{Y}\left\{\int_{-X}^{X}\delta(x)dx\right\}dy \\
&= \lim_{X\to\infty}\lim_{Y\to\infty}\frac{1}{4XY}\int_{-Y}^{Y}1dx \\
&= \lim_{X\to\infty}\lim_{Y\to\infty}\frac{2Y}{4XY} \\
&= \lim_{X\to\infty}\frac{1}{2X} \\
P_\infty(\delta_l) &= 0
\end{aligned}
$$

(c)

$$
\begin{aligned}
E_\infty(e) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\exp[j2\pi(u_0x+v_0y)]|^2\,dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}1\,dxdy \\
E_\infty(e) &= \infty
\end{aligned}
$$

$$
\begin{aligned}
P_\infty(e) &= \lim_{X\to\infty}\lim_{Y\to\infty}\frac{1}{4XY}\int_{-X}^{X}\int_{-Y}^{Y}|\exp[j2\pi(u_0x+v_0y)]|^2\,dxdy \\
&= \lim_{X\to\infty}\lim_{Y\to\infty}\frac{1}{4XY}\int_{-X}^{X}\int_{-Y}^{Y}1\,dxdy \\
P_\infty(e) &= 1
\end{aligned}
$$

(d)

$$
\begin{aligned}
E_\infty(s) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sin^2[2\pi(u_0x+v_0y)]\,dxdy \\
&\overset{②}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1-\cos[4\pi(u_0x+v_0y)]}{2}\,dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2}\,dxdy - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\cos[4\pi(u_0x+v_0y)]}{2}\,dxdy \\
E_\infty(s) &\overset{③}{=} \infty
\end{aligned}
$$

② comes from the trigonometric identity: $\cos(2\theta) = 1 - 2\sin^2(\theta)$.

③ holds because the first integral goes to infinity. The absolute value of the second integral is bounded, although it does not converge as X and Y go to infinity.

$$
\begin{aligned}
P_\infty(s) &= \lim_{X\to\infty}\lim_{Y\to\infty}\frac{1}{4XY}\int_{-X}^{X}\int_{-Y}^{Y}\sin^2[2\pi(u_0x+v_0y)]\,dxdy \\
&= \lim_{X\to\infty}\lim_{Y\to\infty}\frac{1}{4XY}\int_{-Y}^{Y}\left\{\int_{-X}^{X}\frac{1-\cos[4\pi(u_0x+v_0y)]}{2}\,dx\right\}dy
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{X\to\infty}\lim_{Y\to\infty}\frac{1}{4XY}\int_{-Y}^{Y}\left[X+\frac{\sin[4\pi(u_0X+v_0y)]-\sin[4\pi(-u_0X+v_0y)]}{8\pi u_0}\right]dy\\
&\overset{④}{=} \lim_{X\to\infty}\lim_{Y\to\infty}\frac{1}{4XY}\int_{-Y}^{Y}\left[X-\frac{\sin(4\pi u_0X)\cos(4\pi v_0y)}{4\pi u_0}\right]dy\\
&= \lim_{X\to\infty}\lim_{Y\to\infty}\frac{1}{4XY}\left(2XY-\frac{2\sin(4\pi u_0X)\sin(4\pi v_0Y)}{(4\pi)^2u_0v_0}\right)\\
P_\infty(s) &= \frac{1}{2}
\end{aligned}
$$

In order to get ④, we have used trigonometric identity $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$. The rest of the steps are straightforward.

Since $s(x,y)$ is a periodic signal with periods $X_0=1/u_0$ and $Y_0=1/v_0$, we have an alternative way to compute P_∞ by considering only one period in each dimension:

$$
\begin{aligned}
P_\infty(s) &= \frac{1}{4X_0Y_0}\int_{-X_0}^{X_0}\int_{-Y_0}^{Y_0}\sin^2[2\pi(u_0x+v_0y)]\,dxdy\\
&= \frac{1}{4X_0Y_0}\left(2X_0Y_0-\frac{2\sin(4\pi u_0X_0)\sin(4\pi v_0Y_0)}{(4\pi)^2u_0v_0}\right)\\
&= \frac{1}{4X_0Y_0}\left(2X_0Y_0-\frac{2\sin(4\pi)\sin(4\pi)}{(4\pi)^2u_0v_0}\right)\\
P_\infty(s) &= \frac{1}{2}
\end{aligned}
$$

SYSTEMS AND THEIR PROPERTIES

Solution 2.4 Suppose two LSI systems $\mathcal{S}_1$ and $\mathcal{S}_2$ are connected in cascade. For any two input signals $f_1(x,y)$, $f_2(x,y)$, and two constants a_1 and a_2, we have the following:

$$
\begin{aligned}
\mathcal{S}_2[\mathcal{S}_1[a_1f_1(x,y)+a_2f_2(x,y)]] &= \mathcal{S}_2[a_1\mathcal{S}_1[f_1(x,y)]+a_2\mathcal{S}_1[f_2(x,y)]]\\
&= a_1\mathcal{S}_2[\mathcal{S}_1[f_1(x,y)]]+a_2\mathcal{S}_2[\mathcal{S}_1[f_2(x,y)]].
\end{aligned}
$$

So the cascade of two LSI systems is also linear. Now suppose for a given signal $f(x,y)$, we have $\mathcal{S}_1[f(x,y)]=g(x,y)$, and $\mathcal{S}_2[g(x,y)]=h(x,y)$. By using the shift-invariance of the systems, we can prove that the cascade of two LSI systems is also shift invariant:

$$
\mathcal{S}_2[\mathcal{S}_1[f(x-\xi,y-\eta)]]=\mathcal{S}_2[g(x-\xi,y-\eta)]=h(x-\xi,y-\eta).
$$

Eq. (2.47):

$$
\begin{aligned}
g(x,y) &= h_2(x,y)*[h_1(x,y)*f(x,y)]\\
&= h_2(x,y)*\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h_1(\xi,\eta)f(x-\xi,y-\eta)\,d\xi d\eta\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h_2(u,v)\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h_1(\xi,\eta)f(x-u-\xi,y-v-\eta)\,d\xi d\eta\right]dudv\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h_2(u,v)h_1(\xi,\eta)f(x-u-\xi,y-v-\eta)\,d\xi d\eta dudv
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\xi,\eta)\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(u,v) f(x-\xi-u, y-\eta-v)\, du dv\right] d\xi d\eta \\
&= h_1(x,y) * [h_2(x,y) * f(x,y)]
\end{aligned}
$$

By letting $\alpha = u + \xi$, and $\beta = v + \eta$, we have

$$
\begin{aligned}
g(x,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(u,v)h_1(\xi,\eta) f(x-u-\xi, y-v-\eta)\, d\xi d\eta du dv \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(\alpha-\xi,\beta-\eta)h_1(\xi,\eta)\, d\xi d\eta\right] f(x-\alpha, y-\beta)\, d\alpha d\beta \\
&= [h_1(x,y) * h_2(x,y)] * f(x,y)
\end{aligned}
$$

Solution 2.5 1. Suppose the PSF of an LSI system is absolutely integrable.

$$
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |h(x,y)|\, dx dy \le C < \infty \tag{S2.1}
$$

where C is a finite constant. For a bounded input signal $f(x,y)$

$$
|f(x,y)| \le B < \infty, \quad \text{for every } (x,y), \tag{S2.2}
$$

for some finite B, we have

$$
\begin{aligned}
|g(x,y)| &= |h(x,y) * f(x,y)| \\
&= \left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x-\xi, y-\eta) f(\xi,\eta) d\xi d\eta\right| \\
&\le \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |h(x-\xi, y-\eta)| \cdot |f(\xi,\eta)|\, d\xi d\eta \\
&\le B \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |h(x,y)|\, dx dy \\
&\le BC < \infty, \quad \text{for every } (x,y)
\end{aligned} \tag{S2.3}
$$

So $g(x,y)$ is also bounded. The system is BIBO stable.

2. We use contradiction to show that if the LSI system is BIBO stable, its PSF must be absolutely integrable. Suppose the PSF of a BIBO stable LSI system is $h(x,y)$, which is not absolutely integrable, i.e.,

$$
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |h(x,y)|\, dx dy
$$

is not bounded. Then for a bounded input signal $f(x,y) = 1$, the output is

$$
|g(x,y)| = |h(x,y) * f(x,y)| = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |h(x,y)|\, dx dy,
$$

which is also not bounded. So the system can not be BIBO stable. This shows that if the LSI system is BIBO stable, its PSF must be absolutely integrable.

Solution 2.6

(a) If $g'(x, y)$ is the response of the system to input $\sum_{k=1}^{K} w_k f_k(x, y)$, then

$$\begin{aligned} g'(x, y) &= \sum_{k=1}^{K} w_k f_k(x, -1) + \sum_{k=1}^{K} w_k f_k(0, y) \\ &= \sum_{k=1}^{K} w_k [f_k(x, -1) + f_k(0, y)] \\ &= \sum_{k=1}^{K} w_k g_k(x, y) \end{aligned}$$

where $g_k(x, y)$ is the response of the system to input $f_k(x, y)$. Therefore, the system is linear.

(b) If $g'(x, y)$ is the response of the system to input $f(x - x_0, y - y_0)$, then

$$g'(x, y) = f(x - x_0, -1 - y_0) + f(-x_0, y - y_0);$$

while

$$g(x - x_0, y - y_0) = f(x - x_0, -1) + f(0, y - y_0).$$

Since $g'(x, y) \neq g(x - x_0, y - y_0)$, the system is not shift-invariant.

Solution 2.7

(a) If $g'(x, y)$ is the response of the system to input $\sum_{k=1}^{K} w_k f_k(x, y)$, then

$$\begin{aligned} g'(x, y) &= \left(\sum_{k=1}^{K} w_k f_k(x, y)\right)\left(\sum_{k=1}^{K} w_k f_k(x - x_0, y - y_0)\right) \\ &= \sum_{i=1}^{K}\sum_{j=1}^{K} w_i w_j f_i(x, y) f_j(x - x_0, y - y_0), \end{aligned}$$

while

$$\sum_{k=1}^{K} w_k g_k(x, y) = \sum_{k=1}^{K} w_k f_k(x, y) f_k(x - x_0, y - y_0).$$

Since $g'(x, y) \neq \sum_{k=1}^{K} g_k(x, y)$, the system is non-linear.

On the other hand, if $g'(x, y)$ is the response of the system to input $f(x - a, y - b)$, then

$$\begin{aligned} g'(x, y) &= f(x - a, y - b) f(x - a - x_0, y - b - y_0) \\ &= g(x - a, y - b) \end{aligned}$$

and the system is thus shift-invariant.

(b) If $g'(x, y)$ is the response of the system to input $\sum_{k=1}^{K} w_k f_k(x, y)$, then

$$g'(x, y) = \int_{-\infty}^{\infty} \sum_{k=1}^{K} w_k f_k(x, \eta)\, d\eta$$

$$
\begin{aligned}
&= \sum_{k=1}^{K} w_k \left(\int_{-\infty}^{\infty} f_k(x, \eta)\, d\eta \right) \\
&= \sum_{k=1}^{K} w_k g_k(x, y),
\end{aligned}
$$

where $g_k(x, y)$ is the response of the system to input $f_k(x, y)$. Therefore, the system is linear. On the other hand, if $g'(x, y)$ is the response of the system to input $f(x - x_0, y - y_0)$, then

$$
\begin{aligned}
g'(x, y) &= \int_{-\infty}^{\infty} f(x - x_0, \eta - y_0)\, d\eta \\
&= \int_{-\infty}^{\infty} f(x - x_0, \eta - y_0)\, d(\eta - y_0) \\
&= \int_{-\infty}^{\infty} f(x - x_0, \eta)\, d\eta.
\end{aligned}
$$

Since $g(x - x_0, y - y_0) = \int_{-\infty}^{\infty} f(x - x_0, \eta)\, d\eta$, the system is shift-invariant.

Solution 2.8 From the results in Problem 2.5, we know that an LSI system is BIBO stable if and only if its PSF is absolutely integrable.

(a) Not stable. The PSF $h(x, y)$ goes to infinite when x and/or y go to infinity. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |h(x, y)|\, dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x^2+y^2)dxdy = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x^2 dx\right]dy + \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} y^2 dy\right]dx$. Since $\int_{-\infty}^{\infty} x^2 dx = \int_{-\infty}^{\infty} y^2 dy$ is not bounded, then $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x^2 + y^2)dxdy$ is not bounded.

(b) Stable. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |h(x, y)|\, dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\exp\{-(x^2 + y^2)\})dxdy = \left[\int_{-\infty}^{\infty} e^{-x^2} dx\right]^2 = \pi$, which is bounded. So the system is stable.

(c) Not stable. The absolute integral $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 e^{-y^2} dxdy = \int_{-\infty}^{\infty} x^2 \left[\int_{-\infty}^{\infty} e^{-y^2} dy\right] dx = \int_{-\infty}^{\infty} \sqrt{\pi} x^2 dx$ is unbounded. So the system is not stable.

Solution 2.9

(a) $g(x) = \int_{-\infty}^{\infty} f(x - t) f(t) dt$.

(b) Given an input as $af_1(x) + bf_2(x)$, where a, b are some constant, the output is

$$
\begin{aligned}
g'(x) &= [af_1(x) + bf_2(x)] * [af_1(x) + bf_2(x)] \\
&= a^2 f_1(x) * f_1(x) + 2ab f_1(x) * f_2(x) + b^2 f_2(x) * f_2(x) \\
&\neq a g_1(x) + b g_2(x),
\end{aligned}
$$

where $g_1(x)$ and $g_2(x)$ are the output corresponding to an input of $f_1(x)$ and $f_2(x)$ respectively. Hence, the system is nonlinear.

(c) Given a shifted input $f_1(x) = f(x - x_0)$, the corresponding output is

$$
\begin{aligned}
g_1(x) &= f_1(x) * f_1(x) \\
&= \int_{-\infty}^{\infty} f_1(x - t) f_1(t) dt
\end{aligned}
$$

$$= \int_{-\infty}^{\infty} f(x - t - x_0) f_1(t - x_0) dt.$$

Changing variable $t' = t - x_0$ in the above integration, we get

$$\begin{aligned} g_1(x) &= \int_{-\infty}^{\infty} f(x - 2x_0 - t') f_1(t') dt' \\ &= g(x - 2x_0). \end{aligned}$$

Thus, if the input is shifted by x_0, the output is shifted by $2x_0$. Hence, the system is not shift-invariant.

CONVOLUTION OF SIGNALS

Solution 2.10

(a)

$$\begin{aligned} f(x, y)\delta(x - 1, y - 2) &= f(1, 2)\delta(x - 1, y - 2) \\ &= (1 + 2^2)\delta(x - 1, y - 2) \\ &= 5\delta(x - 1, y - 2) \end{aligned}$$

(b)

$$\begin{aligned} f(x, y) * \delta(x - 1, y - 2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi, \eta)\delta(x - \xi - 1, y - \eta - 2)\, d\xi d\eta \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x - 1, y - 2)\delta(x - \xi - 1, y - \eta - 2)\, d\xi d\eta \\ &= f(x - 1, y - 2)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x - \xi - 1, y - \eta - 2)\, d\xi d\eta \\ &= f(x - 1, y - 2) \\ &= (x - 1) + (y - 2)^2 \end{aligned}$$

(c)

$$\begin{aligned} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x - 1, y - 2) f(x, 3) dx dy &\overset{①}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x - 1, y - 2) f(1, 3) dx dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x - 1, y - 2)(1 + 3^2) dx dy \\ &= 10\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x - 1, y - 2) dx dy \\ &\overset{②}{=} 10 \end{aligned}$$

Equality ① comes from the Eq. (2.7) in the text. Equality ② comes from the fact:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x - 1, y - 2) dx dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x, y) dx dy = 1.$$

(d)

$$\begin{aligned}\delta(x-1,y-2)*f(x+1,y+2) &\overset{③}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x-\xi-1,y-\eta-2)f(\xi+1,\eta+2)d\xi d\eta \\ &\overset{④}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x-\xi-1,y-\eta-2)f((x-1)+1,(y-2)+2)d\xi d\eta \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x-\xi-1,y-\eta-2)f(x,y)d\xi d\eta \\ &\overset{⑤}{=} f(x,y)=x+y^2\end{aligned}$$

③ comes from the definition of convolution; ④ comes from the Eq. (2.7) in text; ⑤ is the same as ② in part (c). Alternatively, by using the sifting property of $\delta(x,y)$ and defining $g(x,y)=f(x+1,y+2)$, we have

$$\begin{aligned}\delta(x-1,y-2)*g(x,y) &= g(x-1,y-2) \\ &= f(x-1+1,y-2+2) \\ &= f(x,y) \\ &= x+y^2.\end{aligned}$$

Solution 2.11

(a)

$$\begin{aligned}f(x,y)*g(x,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)g(x-\xi,y-\eta)\,d\xi d\eta \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(\xi)f_2(\eta)g_1(x-\xi)g_2(y-\eta)\,d\xi d\eta \\ f(x,y)*g(x,y) &= \left(\int_{-\infty}^{\infty} f_1(\xi)g_1(x-\xi)\,d\xi\right)\left(\int_{-\infty}^{\infty} f_2(\eta)g_2(y-\eta)\,d\eta\right),\end{aligned}$$

hence their convolution is also separable.

(b)

$$f(x,y)*g(x,y)=(f_1(x)*g_1(x))\,(f_2(y)*g_2(y))\,.$$

Solution 2.12

$$\begin{aligned}g(x,y) &= f(x,y)*h(x,y) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\xi,y-\eta)h(\xi,\eta)d\xi d\eta \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\xi+y-\eta)\exp\{-(\xi^2+\eta^2)\}d\xi d\eta \\ &= (x+y)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\xi^2-\eta^2}d\xi d\eta - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \xi e^{-\xi^2-\eta^2}d\xi d\eta - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \eta e^{-\xi^2-\eta^2}d\xi d\eta \\ &= (x+y)\left[\int_{-\infty}^{\infty} e^{-\xi^2}d\xi\right]^2 - \int_{-\infty}^{\infty} e^{-\eta^2}\left[\int_{-\infty}^{\infty} \xi e^{-\xi^2}d\xi\right]d\eta - \int_{-\infty}^{\infty} e^{-\xi^2}\left[\int_{-\infty}^{\infty} \eta e^{-\eta^2}d\eta\right]d\xi\end{aligned}$$

$$= \pi(x+y) \tag{S2.4}$$

We get (S2.4) by noticing that

$$\int_{-\infty}^{\infty} \xi e^{-\xi^2} d\xi = 0$$

since ξ is an odd function and $e^{-\xi^2}$ is an even function. Also,

$$\int_{-\infty}^{\infty} e^{-\xi^2} d\xi = \sqrt{\pi}.$$

FOURIER TRANSFORMS AND THEIR PROPERTIES

Solution 2.13

(a) See the solution to part **(b)** below. The Fourier transform is

$$\mathcal{F}_2\{\delta_s(x,y)\} = \delta_s(u,v)$$

(b)

$$\mathcal{F}_2\{\delta_s(x,y;\Delta x,\Delta y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta_s(x,y;\Delta x,\Delta y) e^{-j2\pi(ux+vy)}\,dx\,dy$$

$\delta_s(x,y;\Delta x,\Delta y)$ is a periodic signal with periods Δx and Δy in x and y axes. Therefore it can be written as a Fourier series expansion (Please review Oppenheim, A.V.,Willsky, A. S., and Nawad, S. H., *Signals and Systems* for the definition of *Fourier series expansion* of periodic signals.):

$$\delta_s(x,y;\Delta x,\Delta y) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} C_{mn} e^{j2\pi\left(\frac{mx}{\Delta x}+\frac{ny}{\Delta y}\right)}$$

where

$$\begin{aligned} C_{mn} &= \frac{1}{\Delta x \Delta y}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}}\int_{-\frac{\Delta y}{2}}^{\frac{\Delta y}{2}} \delta_s(x,y;\Delta x,\Delta y) e^{-j2\pi\left(\frac{mx}{\Delta x}+\frac{ny}{\Delta y}\right)}\,dx\,dy \\ &= \frac{1}{\Delta x \Delta y}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}}\int_{-\frac{\Delta y}{2}}^{\frac{\Delta y}{2}} \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \delta(x-m\Delta x, y-n\Delta y) e^{-j2\pi\left(\frac{mx}{\Delta x}+\frac{ny}{\Delta y}\right)}\,dx\,dy. \end{aligned}$$

In the integration region $-\frac{\Delta x}{2} < x < \frac{\Delta x}{2}$ and $-\frac{\Delta y}{2} < y < \frac{\Delta y}{2}$ there is only one impulse corresponding to $m=0, n=0$. Therefore, we have:

$$\begin{aligned} C_{mn} &= \frac{1}{\Delta x \Delta y}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}}\int_{-\frac{\Delta y}{2}}^{\frac{\Delta y}{2}} \delta(x,y) e^{-j2\pi\left(\frac{0\cdot x}{\Delta x}+\frac{0\cdot y}{\Delta y}\right)}\,dx\,dy \\ &= \frac{1}{\Delta x \Delta y}. \end{aligned}$$

We have:

$$\delta_s(x, y; \Delta x, \Delta y) = \frac{1}{\Delta x \Delta y} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{j2\pi\left(\frac{mx}{\Delta x} + \frac{ny}{\Delta y}\right)}.$$

Therefore,

$$\begin{aligned}
\mathcal{F}_2\{\delta_s\} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta_s(x, y; \Delta x, \Delta y) e^{-j2\pi(ux+vy)} dx dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\Delta x \Delta y} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{j2\pi\left(\frac{mx}{\Delta x} + \frac{ny}{\Delta y}\right)} e^{-j2\pi(ux+vy)} dx dy \\
&= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{\Delta x \Delta y} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j2\pi\left(\frac{mx}{\Delta x} + \frac{ny}{\Delta y}\right)} e^{-j2\pi(ux+vy)} dx dy \\
&= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{\Delta x \Delta y} \mathcal{F}_2\left\{ e^{j2\pi\left(\frac{mx}{\Delta x} + \frac{ny}{\Delta y}\right)} \right\} \\
&= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{\Delta x \Delta y} \delta\left(u - \frac{m}{\Delta x}, v - \frac{n}{\Delta y}\right) \\
&\overset{⑤}{=} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{\Delta x \Delta y} \cdot \Delta x \Delta y \delta(u\Delta x - m, v\Delta y - n) \\
\mathcal{F}_2\{\delta_s\} &= \delta_s(u\Delta x, v\Delta y)
\end{aligned}$$

Equality ⑤ comes from the property $\delta(ax) = \frac{1}{|a|}\delta(x)$.

(c)

$$\begin{aligned}
\mathcal{F}_2\{s(x, y)\} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(x, y) e^{-j2\pi(ux+vy)} dx dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sin[2\pi(u_0 x + v_0 y)] e^{-j2\pi(ux+vy)} dx dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2j} \left[e^{j2\pi(u_0 x + v_0 y)} - e^{-j2\pi(u_0 x + v_0 y)} \right] e^{-j2\pi(ux+vy)} dx dy \\
&= \frac{1}{2j} \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j2\pi(u_0 x + v_0 y)} e^{-j2\pi(ux+vy)} dx dy \right. \\
&\qquad \left. - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j2\pi(u_0 x + v_0 y)} e^{-j2\pi(ux+vy)} dx dy \right] \\
&= \frac{1}{2j} \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j2\pi[(u-u_0)x + (v-v_0)y]} dx dy \right. \\
&\qquad \left. - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j2\pi[(u+u_0)x + (v+v_0)y]} dx dy \right] \\
\mathcal{F}_2\{s(x, y)\} &= \frac{1}{2j} \left[\delta(u - u_0, v - v_0) - \delta(u + u_0, v + v_0) \right].
\end{aligned}$$

We used Eq. (2.70) twice to get the last equality.

(d)

$$
\begin{aligned}
\mathcal{F}_2(c)(u,v) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c(x,y)e^{-j2\pi(ux+vy)}\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cos[2\pi(u_0x+v_0y)]e^{-j2\pi(ux+vy)}\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2}[e^{j2\pi(u_0x+v_0y)} + e^{-j2\pi(u_0x+v_0y)}]e^{-j2\pi(ux+vy)}\,dx\,dy \\
&= \frac{1}{2}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j2\pi(u_0x+v_0y)}e^{-j2\pi(ux+vy)}\,dx\,dy \right.\\
&\qquad \left. + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j2\pi(u_0x+v_0y)}e^{-j2\pi(ux+vy)}\,dx\,dy\right] \\
&= \frac{1}{2}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j2\pi[(u-u_0)x+(v-v_0)y]}\,dx\,dy \right.\\
&\qquad \left. + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j2\pi[(u+u_0)x+(v+v_0)y]}\,dx\,dy\right] \\
\mathcal{F}_2(c)(u,v) &= \frac{1}{2}[\delta(u-u_0, v-v_0) + \delta(u+u_0, v+v_0)].
\end{aligned}
$$

We used Eq. (2.70) twice to get the last equality.

(e)

$$
\begin{aligned}
\mathcal{F}_2(f)(u,v) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2}e^{-(x^2+y^2)/2\sigma^2}e^{-j2\pi(ux+vy)}\,dx\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2}e^{-(x^2+j4\pi\sigma^2ux)/2\sigma^2}e^{-(y^2+j4\pi\sigma^2vy)/2\sigma^2}\,dx\,dy \\
&= \left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x^2+j4\pi\sigma^2ux)/2\sigma^2}\,dx\right]\left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}}e^{-(y^2+j4\pi\sigma^2vy)/2\sigma^2}\,dy\right] \\
&= \left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x+j2\pi\sigma^2u)^2/2\sigma^2}e^{(j2\pi\sigma^2u)^2/2\sigma^2}\,dx\right]\cdot \\
&\quad \left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}}e^{-(y+j2\pi\sigma^2v)^2/2\sigma^2}e^{(j2\pi\sigma^2v)^2/2\sigma^2}\,dy\right] \\
&= \left[e^{-2\pi^2\sigma^2u^2}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x+j2\pi\sigma^2u)^2/2\sigma^2}\,dx\right]\cdot \\
&\quad \left[e^{-2\pi^2\sigma^2v^2}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}}e^{-(y+j2\pi\sigma^2v)^2/2\sigma^2}\,dy\right] \\
&= e^{-2\pi^2\sigma^2u^2}\cdot e^{-2\pi^2\sigma^2v^2} \\
\mathcal{F}_2(f)(u,v) &= e^{-2\pi^2\sigma^2(u^2+v^2)}.
\end{aligned}
$$

Solution 2.14 The Fourier transform of $f(x)$ is

$$F(u) = \int_{-\infty}^{\infty} f(x)e^{-j2\pi ux}dx.$$

(a)

$$\begin{aligned}
F^*(u) &= \int_{-\infty}^{\infty} \left[f(x)e^{-j2\pi ux}\right]^* dx \\
&= \int_{-\infty}^{\infty} f^*(x)e^{j2\pi ux}dx \\
&= \int_{-\infty}^{\infty} f^*(-\xi)e^{-j2\pi u\xi}d\xi, \text{ let } \xi = -x \\
&= \int_{-\infty}^{\infty} f(\xi)e^{-j2\pi u\xi}d\xi, \text{ since } f(-x) = f(x) \text{ and } f(x) \text{ real} \\
&= F(u)
\end{aligned}$$

(b) Similarly,

$$\begin{aligned}
F^*(u) &= \int_{-\infty}^{\infty} f^*(-\xi)e^{-j2\pi u\xi}d\xi \\
&= \int_{-\infty}^{\infty} -f(\xi)e^{-j2\pi u\xi}d\xi, \text{ since } f(-x) = -f(x) \\
&= -F(u)
\end{aligned}$$

Solution 2.15 In deriving the symmetric property $F^*(u) = F(u)$, we have used the fact that $f(x)$ is real. If $f(x)$ is a complex signal, we have $f^*(-\xi) = f^*(\xi)$, instead of $f^*(-\xi) = f(\xi)$:

$$\begin{aligned}
F^*(u) &= \int_{-\infty}^{\infty} \left[f(x)e^{-j2\pi ux}\right]^* dx \\
&= \int_{-\infty}^{\infty} f^*(-\xi)e^{-j2\pi u\xi}d\xi, \text{ let } \xi = -x \\
&= \int_{-\infty}^{\infty} f^*(\xi)e^{-j2\pi u\xi}d\xi, \\
&= \mathcal{F}\left\{f^*(x)\right\}
\end{aligned}$$

Solution 2.16

(a) Conjugate property: $\mathcal{F}_2(f^*)(u, v) = F^*(-u, -v)$.

$$\begin{aligned}
\mathcal{F}_2(f^*)(u, v) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^*(x, y)e^{-j2\pi(ux+vy)}dxdy \\
&= \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)e^{j2\pi(ux+vy)}dxdy\right]^*
\end{aligned}$$

$$
\begin{aligned}
&= \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi[(-u)x+(-v)y]}dxdy\right]^* \\
&= [F(-u,-v)]^* \\
\mathcal{F}_2(f^*)(u,v) &= F^*(-u,-v)
\end{aligned}
$$

Conjugate symmetry property: If $f(x,y)$ is real, $F(u,v) = F^*(-u,-v)$.
Since $f(x,y)$ is real, $f^*(x,y) = f(x,y)$. Therefore,

$$F^*(-u,-v) = \mathcal{F}_2\{f^*(x,y)\} = \mathcal{F}_2\{f(x,y)\} = F(u,v).$$

(b) Scaling property: $\mathcal{F}_2(f^{ab})(u,v) = \frac{1}{|ab|}\mathcal{F}_2(f)\left(\frac{u}{a},\frac{v}{b}\right)$.

$$
\begin{aligned}
\mathcal{F}_2(f^{ab})(u,v) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(ax,by)e^{-j2\pi(ux+vy)}dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(ax,by)e^{-j2\pi[u(ax)/a+v(by)/b]}\frac{1}{ab}d(ax)d(by) \\
&= \frac{1}{|ab|}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(p,q)e^{-j2\pi[(u/a)p+(v/b)q]}dpdq \\
\mathcal{F}_2(f^{ab})(u,v) &= \frac{1}{|ab|}\mathcal{F}_2(f)\left(\frac{u}{a},\frac{v}{b}\right).
\end{aligned}
$$

(c) Convolution property: $\mathcal{F}_2(f*g)(u,v) = \mathcal{F}_2(g)(u,v)\cdot\mathcal{F}_2(f)(u,v)$.

$$\mathcal{F}_2(f*g)(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)g(x-\xi,y-\eta)d\xi d\eta\right]e^{-j2\pi(ux+vy)}dxdy.$$

Interchange the order of integration, we get

$$
\begin{aligned}
\mathcal{F}_2(f*g)(u,v) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x-\xi,y-\eta)e^{-j2\pi(ux+vy)}dxdy\right]d\xi d\eta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x-\xi,y-\eta)\right. \\
&\qquad \left. e^{-j2\pi[u(x-\xi)+v(y-\eta)]}e^{-j2\pi(u\xi+v\eta)}dxdy\right]d\xi d\eta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)e^{-j2\pi(u\xi+v\eta)}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x-\xi,y-\eta)\right. \\
&\qquad \left. e^{-j2\pi[u(x-\xi)+v(y-\eta)]}dxdy\right]d\xi d\eta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)e^{-j2\pi(u\xi+v\eta)}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(p,q)e^{-j2\pi[up+vq]}dpdq\right]d\xi d\eta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)e^{-j2\pi(u\xi+v\eta)}\mathcal{F}_2(g)(u,v)\,d\xi d\eta \\
&= \mathcal{F}_2(g)(u,v)\cdot\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)e^{-j2\pi(u\xi+v\eta)}\,d\xi d\eta
\end{aligned}
$$

$$\mathcal{F}_2(f * g)(u,v) = \mathcal{F}_2(g)(u,v) \cdot \mathcal{F}_2(f)(u,v).$$

(d) Product property: $\mathcal{F}_2(fg)(u,v) = F(u,v) * G(u,v)$.

$$\begin{aligned}
\mathcal{F}_2(fg)(u,v) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)g(x,y)e^{-j2\pi(ux+vy)}dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\xi,\eta)e^{j2\pi(x\xi+y\eta)}d\xi d\eta\right] f(x,y)e^{-j2\pi(ux+vy)}dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\xi,\eta)\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{j2\pi(x\xi+y\eta)}e^{-j2\pi(ux+vy)}dxdy\right] d\xi d\eta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\xi,\eta)\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi[(u-\xi)x+(v-\eta)y]}dxdy\right] d\xi d\eta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\xi,\eta)F(u-\xi, v-\eta)\, d\xi d\eta \\
&= F(u,v) * G(u,v).
\end{aligned}$$

Solution 2.17 Since both the rect and sinc functions are separable, it is sufficient to show the result for 1-D rect and sinc functions. A 1-D rect function is

$$\text{rect}(x) = \begin{cases} 1, & \text{for } |x| < \frac{1}{2} \\ 0, & \text{for } |x| > \frac{1}{2} \end{cases}$$

$$\begin{aligned}
\mathcal{F}\{\text{rect}(x)\} &= \int_{-\infty}^{\infty} \text{rect}(x)e^{-j2\pi ux}dx \\
&= \int_{-1/2}^{1/2} e^{-j2\pi ux}dx \\
&= \int_{-1/2}^{1/2} \cos(2\pi ux)dx - j\int_{-1/2}^{1/2} \sin(2\pi ux)dx, \quad e^{j\theta} = \cos\theta + j\sin\theta \\
&= \int_{-1/2}^{1/2} \cos(2\pi ux)dx \\
&= \frac{\sin(\pi u)}{\pi u} \\
&= \text{sinc}(u)
\end{aligned}$$

Therefore, we have $\mathcal{F}\{\text{sinc}(x)\} = \text{rect}(u)$.

Using Parseval's Theorem, we have

$$\begin{aligned}
E_\infty &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \| \text{rect}(x,y)\|^2 dxdy \\
&= \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} dxdy
\end{aligned}$$

$$= 1$$

For the sinc function, $P_\infty = 0$, because E_∞ is finite.

Solution 2.18 Since the signal is separable, we have

$$\mathcal{F}[f(x,y)] = \mathcal{F}_{1D}[\sin(2\pi ax)]\mathcal{F}_{1D}[\cos(2\pi by)].$$

$$\mathcal{F}_{1D}[\sin(2\pi ax)] = \frac{1}{2j}\left[\delta(u-a) - \delta(u+a)\right]$$

$$\mathcal{F}_{1D}[\cos(2\pi by)] = \frac{1}{2}\left[\delta(v-b) + \delta(v+b)\right]$$

So,

$$\mathcal{F}[f(x,y)] = \frac{1}{4j}\left[\delta(u-a)\delta(v-b) - \delta(u+a)\delta(v-b) + \delta(u-a)\delta(v+b) - \delta(u+a)\delta(v+b)\right].$$

Now we need to show that $\delta(u)\delta(v) = \delta(u,v)$ (in a generalized way):

$$\delta(u)\delta(v) = 0, \quad \text{for } u \neq 0, \text{ or } v \neq 0$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u,v)\delta(u)\delta(v)dudv = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(u,v)\delta(u)du\right]\delta(v)dv = \int_{-\infty}^{\infty} f(0,v)\delta(v)dv = f(0,0)$$

Based on the argument above $\delta(u)\delta(v) = \delta(u,v)$, and

$$\mathcal{F}[f(x,y)] = \frac{1}{4j}\left[\delta(u-a,v-b) - \delta(u+a,v-b) + \delta(u-a,v+b) - \delta(u+a,v+b)\right].$$

The above solution can also be obtained by using the relationship:

$$\sin(2\pi ax)\cos(2\pi by) = \frac{1}{2}\left[\sin(2\pi(ax-by)) + \sin(2\pi(ax+by))\right].$$

Solution 2.19 A function $f(x,y)$ can be expressed in polar coordinates as:

$$f(x,y) = f(r\cos\theta, r\sin\theta) = f_P(r,\theta).$$

If it is circularly symmetric, we have $f_p(r,\theta)$ is constant for fixed r. The Fourier transform of $f(x,y)$ is defined as:

$$\begin{aligned} F(u,v) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy \\ &= \int_0^{\infty}\int_0^{2\pi} f_p(r,\theta)e^{-j2\pi(ur\cos\theta+vr\sin\theta)}rdrd\theta \\ &= \int_0^{\infty} f_p(r,\theta)\left[\int_0^{2\pi} e^{-j2\pi(ur\cos\theta+vr\sin\theta)}d\theta\right]rdr. \end{aligned}$$

Letting $u = q\cos\phi$ and $v = q\sin\phi$, the above equation becomes:

$$F(u, v) = \int_0^{\infty} f_p(r, \theta)\left[\int_0^{2\pi} e^{-j2\pi qr\cos(\phi-\theta)}d\theta\right] rdr.$$

Since $F(u, v)$ is also circularly symmetric, it can be written as $F_q(q, \phi)$ and is constant for fixed q. In particular, $F_q(q, \phi) = F_q(q, \pi/2)$, and therefore

$$F_q(q, \phi) = F_q(q, \pi/2) = \int_0^{\infty} f_p(r, \theta)\left[\int_0^{2\pi} e^{-j2\pi qr\sin\theta}d\theta\right] rdr.$$

Now we will show that (2.109) holds:

$$\begin{aligned}
& \int_0^{2\pi} e^{-j2\pi qr\sin\theta}d\theta \\
= & \int_0^{2\pi} \cos(2\pi qr\sin\theta)d\theta - j\int_0^{2\pi} \sin(2\pi qr\sin\theta) \\
\overset{①}{=} & \ 2\int_0^{\pi} \cos(2\pi qr\sin\theta)d\theta \\
= & \ 2\pi J_0(2\pi qr)
\end{aligned}$$

Equality ① holds because $\cos(-\theta) = \cos(\theta)$, and $\sin(\theta) = -\sin(\theta)$.

Based on the above derivation, we have (2.109).

Solution 2.20 The unit disk is expressed as $f(r) = \mathrm{rect}(r)$, its Hankel transform is:

$$\begin{aligned}
F(q) &= 2\pi\int_0^{\infty} f(r)J_0(2\pi qr)rdr \\
&= 2\pi\int_0^{\infty} \mathrm{rect}(r)J_0(2\pi qr)rdr \\
&= 2\pi\int_0^{1/2} J_0(2\pi qr)rdr
\end{aligned}$$

Change of variable

$$\begin{aligned}
s &= 2\pi qr \\
r &= \frac{s}{2\pi q} \\
dr &= \frac{ds}{2\pi q}
\end{aligned}$$

Apply the change of variable to the above equation

$$F(q) = \frac{1}{2\pi q^2}\int_0^{\pi q} J_0(s)sds$$

Note that

$$\int_0^x J_0(\epsilon)\epsilon d\epsilon = xJ_1(x)$$

So

$$\begin{aligned} F(q) &= \frac{J_1(\pi q)}{2q} \\ &= \text{jinc}(q) \end{aligned}$$

TRANSFER FUNCTION

Solution 2.21

(a) The impulse response function is shown in Figure S2.1.

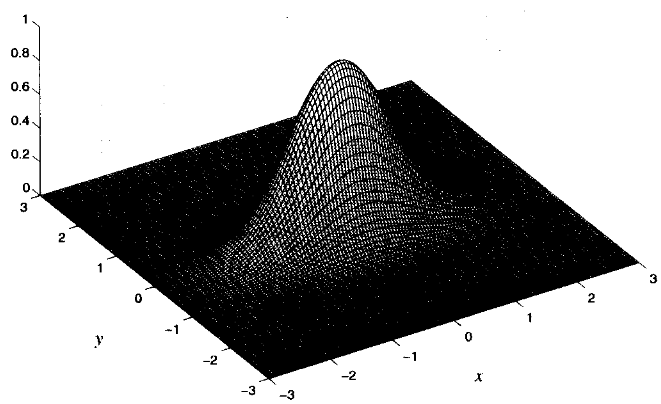

Figure S2.1 Impulse response function of the system. [Problem 2.21(a)]

(b) The transfer function of the function is the Fourier transform of the impulse response function:

$$\begin{aligned} H(u,v) &= \mathcal{F}\{h(x,y)\} \\ &= \mathcal{F}\{e^{-\pi x^2}\}\mathcal{F}\{e^{-\pi y^2/4}\}, \text{ since } h(x,y) \text{ is separable} \\ &= 4e^{-\pi(u^2+4v^2)}. \end{aligned}$$

Solution 2.22

(a) The 1D profile of the bar phantom is:

$$f(x) = \begin{cases} 1, & \frac{k-1}{2}w \le x \le \frac{k+1}{2}w \\ 0, & \frac{k+1}{2}w \le x \le \frac{k+3}{2}w \end{cases},$$

where k is an integer.

The response of the system to the bar phantom is:

$$g(x) = f(x) * l(x) = \int_{-\infty}^{\infty} f(x-\xi)l(\xi)d\xi.$$

At the center of the bar, we have

$$\begin{aligned} g(0) &= \int_{-\infty}^{\infty} f(0-\xi)l(\xi)d\xi \\ &= \int_{-w/2}^{w/2} \cos(\alpha\xi)d\xi \\ &= \frac{2}{\alpha}\sin\left(\frac{\alpha w}{2}\right). \end{aligned}$$

At the point halfway between two adjacent bars, we have

$$\begin{aligned} g(w) &= \int_{-\infty}^{\infty} f(w-\xi)l(\xi)d\xi \\ &= \int_{w-\pi/2\alpha}^{w/2} \cos(\alpha\xi)d\xi + \int_{3w/2}^{w+\pi/2\alpha} \cos(\alpha\xi)d\xi \\ &= 2\int_{w-\pi/2\alpha}^{w/2} \cos(\alpha\xi)d\xi \\ &= \frac{2}{\alpha}\left[\sin\left(\frac{\alpha w}{2}\right) - \sin\left(\alpha w - \frac{\pi}{2}\right)\right]. \end{aligned}$$

(b) From the line spread function alone, we cannot tell whether the system is isotropic. The line spread function is a "projection" of the point spread function. During the projection, the information along the y-direction is lost.

(c) Since the system is separable with $h(x, y) = h_{1D}(x)h_{1D}(y)$, we know that

$$\begin{aligned} l(x) &= \int_{-\infty}^{\infty} h(x,y)dy \\ &= h_{1D}(x)\int_{-\infty}^{\infty} h_{1D}(y)dy \end{aligned}$$

So $h_{1D}(x) = cl(x)$ where $1/c = \int_{-\infty}^{\infty} h_{1D}(y)dy$. Hence,

$$\begin{aligned} 1/c &= \int_{-\infty}^{\infty} cl(y)dy \\ 1/c^2 &= \int_{-\pi/2\alpha}^{\pi/2\alpha} \cos(\alpha y)dy \\ 1/c^2 &= 2/\alpha \end{aligned}$$

Therefore,

$$h(x,y) = \begin{cases} \dfrac{\alpha}{2}\cos(\alpha x)\cos(\alpha y) & |\alpha x| \le \pi/2 \text{ and } |\alpha y| \le \pi/2 \\ 0 & \text{otherwise} \end{cases}.$$

The transfer function is

$$\begin{aligned}
H(u,v) &= \mathcal{F}_{2D}\{h(x,y)\} \\
&= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h(x,y)e^{j2\pi ux}dx\right]e^{j2\pi uy}dy \\
&= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h_{1D}(x)h_{1D}(y)e^{j2\pi ux}dx\right]e^{j2\pi uy}dy \\
&= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h_{1D}(x)e^{j2\pi ux}dx\right]h_{1D}(y)e^{j2\pi uy}dy \\
&= \int_{-\infty}^{\infty} h_{1D}(x)e^{j2\pi ux}dx \int_{-\infty}^{\infty} h_{1D}(y)e^{j2\pi uy}dy \\
&= H_{1D}(u)H_{1D}(v)
\end{aligned}$$

which is also separable with $H(u,v) = H_{1D}(u)H_{1D}(v)$.

$$\begin{aligned}
H_{1D} &= \sqrt{\frac{\alpha}{2}}\mathcal{F}_{1D}\{l(x)\} \\
&= \sqrt{\frac{\alpha}{2}}\mathcal{F}_{1D}\{\cos(\alpha x)\} * \mathcal{F}_{1D}\left\{\text{rect}\left(\frac{\alpha x}{\pi}\right)\right\} \\
&= \sqrt{\frac{\pi}{2}}\left[\text{sinc}\left(\frac{\pi}{\alpha}(u-\alpha/2\pi)\right) + \text{sinc}\left(\frac{\pi}{\alpha}(u+\alpha/2\pi)\right)\right].
\end{aligned}$$

The transfer function is

$$\begin{aligned}
H(u,v) = \frac{\pi}{2}&\left[\text{sinc}\left(\frac{\pi}{\alpha}(u-\alpha/2\pi)\right) + \text{sinc}\left(\frac{\pi}{\alpha}(u+\alpha/2\pi)\right)\right] \\
&\left[\text{sinc}\left(\frac{\pi}{\alpha}(v-\alpha/2\pi)\right) + \text{sinc}\left(\frac{\pi}{\alpha}(v+\alpha/2\pi)\right)\right].
\end{aligned}$$

SAMPLING THEORY

Solution 2.23

(a)

$$\begin{aligned}
f_s(t) &= f(t)\delta_s(t;\Delta T) \\
&= \sum_{m=-\infty}^{\infty} f(t)\delta(t-m\Delta T) \\
&= \sum_{m=-\infty}^{\infty} f(m\Delta T)\delta(t-m\Delta T)
\end{aligned}$$

Since

$$f(t) = \begin{cases} \sin\left(\frac{2\pi t}{T}\right), & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases}.$$

and $\Delta T = 0.25T$, we have

$$f_s(t) = \delta(t - 0.25T) - \delta(t - 0.75T).$$

$$f_d(m) = f(m\Delta T) = \begin{cases} 1, & m = 1 \\ -1, & m = 3 \\ 0, & \text{otherwise} \end{cases}.$$

(b) The signal $f_h(t)$ is referred to as a *zero-order hold*. By definition,

$$\begin{aligned} f_h(t) &= \begin{cases} 1, & 0.25T \le t < 0.5T \\ -1, & 0.75T \le t < T \\ 0, & \text{otherwise} \end{cases} \\ &= \text{rect}\left(\frac{t - 0.375T}{0.5T}\right) - \text{rect}\left(\frac{t - 0.825T}{0.5T}\right). \end{aligned}$$

Using the properties of the Fourier transform, we have

$$\begin{aligned} F_h(f) &= \mathcal{F}(f_h(t)) \\ &= \int_{-\infty}^{\infty} f_h(t)e^{-j2\pi f t}dt \\ &= 0.25T\,\text{sinc}(0.25Tf)e^{-j2\pi(0.375Tf)} - 0.25T\,\text{sinc}(0.25Tf)e^{-j2\pi(0.825Tf)} \\ &= 0.25T\,\text{sinc}(0.25Tf)\left[e^{-j2\pi(0.375Tf)} - e^{-j2\pi(0.825Tf)}\right] \end{aligned}$$

(c) For $\Delta T = 0.5T$, we have

$$\begin{aligned} f_s(t) &= 0 \\ f_d(m) &= 0 \\ f_h(t) &= 0 \\ F_h(f) &= 0. \end{aligned}$$

Solution 2.24 Since the Nyquist sampling periods for 1-D band-limited signals $f(x)$ and $g(x)$ are Δ_f and Δ_g, the highest frequency of $f(x)$ and $g(x)$ are $\frac{1}{2\Delta_f}$ and $\frac{1}{2\Delta_g}$. In order to find the Nyquist sampling periods, we need to find the highest frequency for each of the signals.

(a) A shift in location does not change the frequency components of a signal, so the magnitude spectrum of $f(x - x_0)$ is the same as that of $f(x)$. The Nyquist sampling period of $f(x - x_0)$ is Δ_f.

(b) The Fourier transform of $f(x)+g(x)$ is $\mathcal{F}[f(x)+g(x)] = \mathcal{F}[f(x)]+\mathcal{F}[g(x)]$. The highest frequency of $f(x) + g(x)$ is $\max(\frac{1}{2\Delta_f}, \frac{1}{2\Delta_g})$, so the Nyquist sampling period of $f(x) + g(x)$ is $\min(\Delta_f, \Delta_g)$.

(c) The Fourier transform of $f(x) * f(x)$ is $\mathcal{F}[f(x)]^2$. The highest frequency of $f(x) * f(x)$ is $\frac{1}{2\Delta_f}$, The Nyquist sampling period of $f(x) * f(x)$ is Δ_f.

(d) The Fourier transform of $f(x)g(x)$ is $\mathcal{F}[f(x)] * \mathcal{F}[g(x)]$, The highest frequency of $f(x)g(x)$ is $\frac{1}{2\Delta_f} + \frac{1}{2\Delta_g}$, and the Nyquist sampling period is $\frac{\Delta_f \Delta_g}{\Delta_f + \Delta_g}$.

(e) If $f(x) \ge 0$, $\|f(x)\| = f(x)$, the Nyquist sampling period of $\|f(x)\|$ is Δ_f. But in general, the operation of taking absolute value will reverse part of the original signal $f(x)$, and therefore introduce high frequency component. In general case, $\|f(x)\|$ is no more band-limit signal, even though $f(x)$ is.

Solution 2.25 The sampling frequencies are $\frac{1}{\Delta x} = 1.5$ and $\frac{1}{\Delta y} = 1.5$. From the Sampling Theorem, in order to avoid aliasing the cutoff frequencies of the low-pass filtered signal $f * h$ must satisfy:

$$U \le \frac{1}{2\Delta x} = 0.75, \text{ and } V \le \frac{1}{2\Delta y} = 0.75.$$

Thus, the ideal low-pass filter $h(x, y)$ that gives the maximum possible frequency content must have a frequency response as

$$H(u, v) = \begin{cases} 1, \text{ if } |u| \le 0.75 \text{ and } |v| \le 0.75 \\ 0, \text{ otherwise} \end{cases}.$$

$H(u, v)$ is one inside a square region and zero outside.

The PSF of the required anti-aliasing low-pass filter can be computed as:

$$\begin{aligned} h(x, y) &= \mathcal{F}_2^{-1}(H(u, v)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(u, v)e^{j2\pi(ux+vy)}\,du\,dv \\ &= \int_{-0.75}^{0.75}\int_{-0.75}^{0.75} e^{j2\pi(ux+vy)}\,du\,dv \\ &= \left(\int_{-0.75}^{0.75} e^{j2\pi ux}\,du\right)\cdot\left(\int_{-0.75}^{0.75} e^{j2\pi vy}\,dv\right) \\ &= \frac{\exp[j2\pi(0.75)x] - \exp[j2\pi(-0.75)x]}{j2\pi x}\cdot\frac{\exp[j2\pi(0.75)y] - \exp[j2\pi(-0.75)y]}{j2\pi y} \\ &= \frac{\sin(1.5\pi x)}{\pi x}\cdot\frac{\sin(1.5\pi y)}{\pi y}. \end{aligned}$$

From Table 2.1, we know that

$$\mathcal{F}_2(f)(u, v) = e^{-\pi(u^2+v^2)}.$$

Thus, the total spectrum energy of $f(x, y)$ is

$$\begin{aligned} E_{\text{total}} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\mathcal{F}_2(f)(u, v)|^2\,du\,dv \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi(u^2+v^2)}\,du\,dv \\ &= \left(2\pi\sigma^2\right)\frac{1}{2\pi\sigma^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(u^2+v^2)/2\sigma^2}\,du\,dv \text{ with } \sigma^2 = \frac{1}{4\pi} \\ &= 2\pi\cdot\frac{1}{4\pi} \\ &= 0.5. \end{aligned}$$

The spectrum that is kept by the low pass filter has energy of

$$\begin{aligned} E_{\text{preserve}} &= \int_{-0.75}^{0.75}\int_{-0.75}^{0.75} e^{-2\pi(u^2+v^2)}\,du\,dv \\ &= \int_{-0.75}^{0.75} e^{-2\pi u^2}\,du\cdot\int_{-0.75}^{0.75} e^{-2\pi v^2}\,dv \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{1}{\sqrt{2\pi}} \int_{-0.75\sqrt{2\pi}}^{0.75\sqrt{2\pi}} e^{-t^2}\, dt\right)^2 \\
&= \left[\frac{1}{\sqrt{2\pi}} \frac{\sqrt{\pi}}{2} 2\operatorname{erf}\left(0.75\sqrt{2\pi}\right)\right]^2 \\
&= \frac{1}{2}\left[\operatorname{erf}\left(0.75\sqrt{2\pi}\right)\right]^2 \\
&\approx \frac{1}{2}[0.992]^2 \\
&\approx 0.492,
\end{aligned}
$$

where erf(·) is the error function.

Thus, the percentage of the spectrum energy that is preserved is

$$
\frac{E_{\text{preserve}}}{E_{\text{total}}} = \frac{0.492}{0.5} = 98.4\%
$$

Since the spectrum of $f(x, y)$, which is $\mathcal{F}_2(f)(u, v) = e^{-\pi(u^2+v^2)}$, is non-zero for all $(u, v) \in (-\infty, \infty) \times (-\infty, \infty)$, it is impossible to sample $f(x, y)$ free of aliasing without using an anti-aliasing filter.

APPLICATIONS, EXTENSIONS AND ADVANCED TOPICS

Solution 2.26

(a) The system is separable. $h(x, y) = e^{-(|x|+|y|)} = e^{-|x|}e^{-|y|}$.

(b) The system is not isotropic since $h(x, y)$ is not a function of $r = \sqrt{x^2 + y^2}$.

Additional comments: An easy check is to plug in $x = 1,\ y = 1$ and $x = 0,\ y = \sqrt{2}$ into $h(x, y)$. By noticing that $h(1, 1) \neq h(0, \sqrt{2})$, we can conclude that $h(x, y)$ is not rotationally invariant, and hence not isotropic.

Isotropy is rotational symmetry around a point, not just symmetry about a few axes, for example x- and y- axes. $h(x, y) = e^{-(|x|+|y|)}$ is symmetric about a few lines, but it is not rotationally invariant.

When we studied the properties of Fourier transform, we learned that if a signal is isotropic, than its Fourier transform has a certain symmetry. Note that the symmetry of the Fourier transform is only a necessary, but not sufficient, condition for the signal to be isotropic.

(c) Modulation transformation function of a anisotropic system is defined as

$$
\text{MTF}(u, v) = \frac{H(u, v)}{H(0, 0)}
$$

where $H(u, v)$ is the Fourier transform of $h(x, y)$.

$$
\begin{aligned}
H(u, v) &= |\mathcal{F}_2(h(x, y))| \\
&= \left|\mathcal{F}_2\left(e^{-(|x|+|y|)}\right)\right| \\
&= \left|\mathcal{F}_1\left(e^{-|x|}\right)\right| \cdot \left|\mathcal{F}_1\left(e^{-|y|}\right)\right|
\end{aligned}
$$

$$
\begin{aligned}
&= \mathcal{F}_1\left(e^{-|x|}\right) \cdot \mathcal{F}_1\left(e^{-|y|}\right) \\
&= \frac{2}{1+(2\pi u)^2}\frac{2}{1+(2\pi v)^2} \\
&= \frac{4}{(1+(2\pi u)^2)(1+(2\pi v)^2)}.
\end{aligned}
$$

So the MTF is

$$
\mathrm{MTF}(u,v) = \frac{H(u,v)}{H(0,0)} = \frac{1}{(1+(2\pi u)^2)(1+(2\pi v)^2)}.
$$

(d) The response is

$$
\begin{aligned}
g(x,y) &= h(x,y) * f(x,y) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\xi,\eta) f(x-\xi, y-\eta) d\xi d\eta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(|\xi|+|\eta|)}\delta(x-\xi) d\xi d\eta \\
&= \int_{-\infty}^{\infty} e^{-(|x|+|\eta|)} d\eta \\
&= e^{-|x|}\int_{-\infty}^{\infty} e^{-|\eta|} d\eta \\
&= e^{-|x|}\left[\int_{-\infty}^{0} e^{\eta} d\eta + \int_{0}^{\infty} e^{-\eta} d\eta\right] \\
&= 2e^{-|x|}
\end{aligned}
$$

(e) The response is

$$
\begin{aligned}
g(x,y) &= h(x,y) * f(x,y) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\xi,\eta) f(x-\xi, y-\eta) d\xi d\eta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(|\xi|+|\eta|)}\delta(x-\xi-y+\eta) d\xi d\eta \\
&= \int_{-\infty}^{\infty} e^{-|\eta|}\left[\int_{-\infty}^{\infty} e^{-|\xi|}\delta(x-\xi-y+\eta) d\xi\right] d\eta \\
&= \int_{-\infty}^{\infty} e^{-|\eta|} e^{-|x-y+\eta|} d\eta
\end{aligned}
$$

1. Now assume $x - y < 0$, then $x - y + \eta < \eta$. The range of integration in the above can be divided into three parts (see Fig. S2.2):

I. $\eta \in (-\infty, 0)$. In this interval, $x - y + \eta < \eta < 0$. $|\eta| = -\eta$, $|x - y + \eta| = -(x - y + \eta)$;

II. $\eta \in [0, -(x - y))$. In this interval, $x - y + \eta < 0 \le \eta$. $|\eta| = \eta$, $|x - y + \eta| = -(x - y + \eta)$;

III. $\eta \in [-(x - y), \infty)$. In this interval, $0 \le x - y + \eta < \eta$. $|\eta| = \eta$, $|x - y + \eta| = x - y + \eta$.

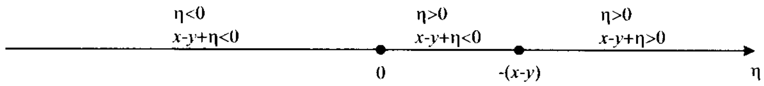

Figure S2.2 For $x - y < 0$, integration interval $(-\infty, \infty)$ can be partitioned into three segments. [Problem 2.26(e)]

Based on the above analysis, we have:

$$\begin{aligned}
g(x, y) &= \int_{-\infty}^{\infty} e^{-|\eta|} e^{-|x-y+\eta|} d\eta \\
&= \int_{-\infty}^{0} e^{-(|\eta|+|x-y+\eta|)} d\eta + \int_{0}^{-(x-y)} e^{-(|\eta|+|x-y+\eta|)} + \int_{-(x-y)}^{\infty} e^{-(|\eta|+|x-y+\eta|)} \\
&= \int_{-\infty}^{0} e^{x-y+2\eta} d\eta + \int_{0}^{-(x-y)} e^{x-y} d\eta + \int_{-(x-y)}^{\infty} e^{-(x-y+2\eta)} d\eta \\
&= \frac{1}{2} e^{x-y} - (x-y) e^{x-y} + \frac{1}{2} e^{x-y} \\
&= [1 - (x-y)] e^{x-y}
\end{aligned}$$

2. For $x - y \geq 0$, $\eta < x - y + \eta$. The range of integration in the above can be divided into three parts (see Fig. S2.3):

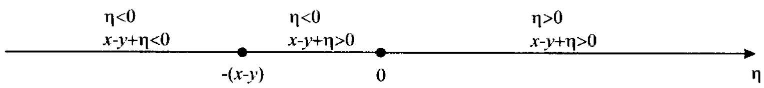

Figure S2.3 For $x - y > 0$, integration interval $(-\infty, \infty)$ can be partitioned into three segments. [Problem 2.26(e)]

I. $\eta \in (-\infty, -(x-y))$. In this interval, $\eta < x - y + \eta < 0$. $|\eta| = -\eta$, $|x - y + \eta| = -(x - y + \eta)$;
II. $\eta \in [-(x-y), 0)$. In this interval, $\eta < 0 \leq x - y + \eta$. $|\eta| = -\eta$, $|x - y + \eta| = x - y + \eta$;
III. $\eta \in [0, \infty)$. In this interval, $0 \leq \eta < x - y + \eta$. $|\eta| = \eta$, $|x - y + \eta| = x - y + \eta$.

$$\begin{aligned}
g(x, y) &= \int_{-\infty}^{\infty} e^{-|\eta|} e^{-|x-y+\eta|} d\eta \\
&= \int_{-\infty}^{-(x-y)} e^{-(|\eta|+|x-y+\eta|)} d\eta + \int_{-(x-y)}^{0} e^{-(|\eta|+|x-y+\eta|)} + \int_{0}^{\infty} e^{-(|\eta|+|x-y+\eta|)} \\
&= \int_{-\infty}^{-(x-y)} e^{x-y+2\eta} d\eta + \int_{-(x-y)}^{0} e^{-(x-y)} d\eta + \int_{0}^{\infty} e^{-(x-y+2\eta)} d\eta \\
&= \frac{1}{2} e^{-(x-y)} + (x-y) e^{-(x-y)} + \frac{1}{2} e^{-(x-y)} \\
&= [1 + (x-y)] e^{-(x-y)}
\end{aligned}$$

Based on the above two steps, we have:

$$g(x, y) = (1 + |x - y|) e^{-|x-y|}$$

Solution 2.27

(a) Yes, it is shift invariant because its impulse response depends on $x - \xi$.

(b) By linearity, the output is

$$g(x) = e^{\frac{-(x+1)^2}{2}} + e^{\frac{-(x)^2}{2}} + e^{\frac{-(x-1)^2}{2}}.$$

(c) We need to find the Fourier transform in order to find the MTF. From Table 2.1, we know that

$$e^{-\pi x^2} \leftrightarrow e^{-\pi u^2}.$$

Using the scaling property of the Fourier transform, we have

$$e^{-x^2/2} = e^{-\pi\left(\frac{x}{\sqrt{2\pi}}\right)^2} \leftrightarrow \sqrt{2\pi}e^{-2\pi^2u^2}.$$

$$\text{MTF}(u) = \frac{H(u)}{|H(0)|} = e^{-2\pi^2u^2}.$$

Solution 2.28

(a) The impulse response of the filter is the inverse Fourier transform of $H(u)$, which can be written as

$$H(u) = 1 - \text{rect}\left(\frac{u}{2U_0}\right)$$

Using the linearity of the Fourier transform and the Fourier transform pairs

$$\begin{aligned}\mathcal{F}\{\delta(t)\} &= 1\\ \mathcal{F}\{\text{sinc}(t)\} &= \text{rect}(u)\end{aligned}$$

We have

$$\begin{aligned}h(t) &= \mathcal{F}^{-1}\{H(u)\}\\ &= \delta(t) - 2U_0\,\text{sinc}(2U_0t)\end{aligned}$$

(b) The system response to $f(t) = c$ is 0, since $f(t)$ contains only DC component while $h(t)$ passes only high frequency components.

$$\begin{aligned}f(t) * h(t) &= f(t) * [\delta(t) - 2U_0\,\text{sinc}(2U_0t)]\\ &= f(t) - 2U_0 f(t) * \text{sinc}(2U_0t)\\ &= c - c\int_{-\infty}^{\infty} 2U_0\,\text{sinc}(2U_0t)dt\\ &= c - c\int_{-\infty}^{\infty} \text{sinc}(\tau)d\tau\\ &= 0\end{aligned}$$

The system response to $f(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$ is

$$f(t) * h(t) = f(t) * [\delta(t) - 2U_0\,\text{sinc}(2U_0t)]$$

$$
\begin{aligned}
&= f(t) - 2U_0 f(t) * \operatorname{sinc}(2U_0 t) \\
&= f(t) - \int_{-\infty}^{\infty} f(x) 2U_0 \operatorname{sinc}(2U_0(t-x))dx \\
&= f(t) - \int_{0}^{\infty} 2U_0 \operatorname{sinc}(2U_0(t-x))dx \\
&= f(t) + \int_{t}^{-\infty} 2U_0 \operatorname{sinc}(2U_0(y))dy \\
&= f(t) - \int_{-\infty}^{t} 2U_0 \operatorname{sinc}(2U_0(y))dy \\
&= \begin{cases} 1 - \int_{-\infty}^{0} 2U_0 \operatorname{sinc}(2U_0(y))dy + \int_{t}^{0} 2U_0 \operatorname{sinc}(2U_0(y))dy & t < 0 \\ 1 - \int_{-\infty}^{0} 2U_0 \operatorname{sinc}(2U_0(y))dy - \int_{0}^{t} 2U_0 \operatorname{sinc}(2U_0(y))dy & t > 0 \end{cases} \\
&= \begin{cases} -\frac{1}{2} + \int_{t}^{0} 2U_0 \operatorname{sinc}(2U_0(y))dy & t < 0 \\ 1 - \frac{1}{2} - \int_{0}^{t} 2U_0 \operatorname{sinc}(2U_0(y))dy & t > 0 \end{cases} \\
&= \begin{cases} -\frac{1}{2} + \int_{t}^{0} 2U_0 \operatorname{sinc}(2U_0(y))dy & t < 0 \\ \frac{1}{2} - \int_{0}^{t} 2U_0 \operatorname{sinc}(2U_0(y))dy & t > 0 \end{cases}
\end{aligned}
$$

Solution 2.29

(a) The rect function is defined as

$$\operatorname{rect}(t) = \begin{cases} 1, & |t| \le 1/2 \\ 0, & \text{otherwise} \end{cases}$$

So we have

$$\operatorname{rect}\left(\frac{t}{T}\right) = \begin{cases} 1, & |t| \le T/2 \\ 0, & \text{otherwise} \end{cases}$$

and

$$\operatorname{rect}\left(\frac{t+0.75T}{0.5T}\right) = \begin{cases} 1, & |t+0.75T| \le T/4 \\ 0, & \text{otherwise} \end{cases}.$$

So,

$$h(t) = \begin{cases} -1/T, & -T < t < -T/2 \\ 1/T, & -T/2 < t < T/2 \\ -1/T, & T/2 < t < T \\ 0, & \text{otherwise} \end{cases}.$$

The impulse response is plotted in Fig. S2.4.

The absolute integral of $h(t)$ is $\int_{-\infty}^{\infty} |h(t)|^2 dt = 2/T$. So The system is stable when $T > 0$. The system is not causal, since $h(t) \neq 0$ for $-T < t < 0$.

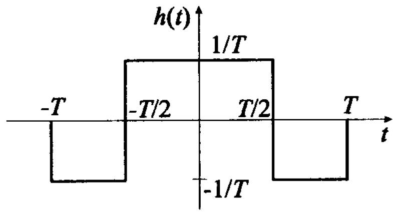

Figure S2.4 The impulse response $h(t)$. [Problem 2.29(a)]

(b) The response of the system to a constant signal $f(t) = c$ is

$$g(t) = f(t) * h(t) = \int_{-\infty}^{\infty} f(t-\tau)h(\tau)d\tau = c\int_{-\infty}^{\infty} h(\tau)d\tau = 0.$$

(c) The response of the system to the unit step function is

$$g(t) = f(t) * h(t) = \int_{-\infty}^{\infty} f(t-\tau)h(\tau)d\tau = \int_{-\infty}^{t} h(\tau)d\tau$$

$$g(t) = \begin{cases} 0, & t < -T \\ -t/T - 1, & -T < t < -T/2 \\ t/T, & -T/2 < t < T/2 \\ -t/T + 1, & T/2 < t < T \\ 0, & t > T \end{cases}$$

The response of the system to the unit step signal is plotted in Figure S2.5.

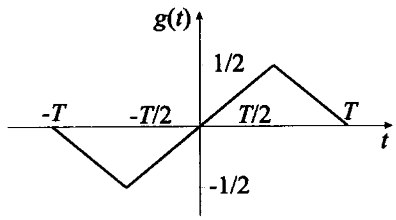

Figure S2.5 The response of the system to the unit step signal. [Problem 2.29(c)]

(d) The Fourier transform of a rect function is a sinc function (see Problem 2.17). By using the properties of the Fourier transform (scaling, shifting, and linearity), we have,

$$\begin{aligned} H(u) &= \mathcal{F}\{h(t)\} \\ &= -0.5e^{-j2\pi u(-0.75T)}\,\mathrm{sinc}(0.5uT) + \mathrm{sinc}(uT) - 0.5e^{-j2\pi u(0.75T)}\,\mathrm{sinc}(0.5uT) \\ &= \mathrm{sinc}(uT) - \cos(1.5\pi uT)\,\mathrm{sinc}(0.5uT) \end{aligned}$$

(e) The magnitude spectrum of $h(t)$ is plotted in Figure S2.6.

(f) From the calculation in part **(d)** and the plot in part **(c)**, it can be seen that $|H(0)| = 0$. So the output of the system does not have a DC component. The system is not a low pass filter. The system is not

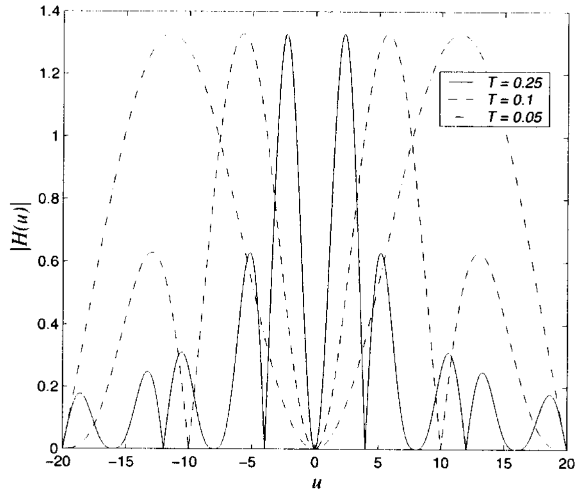

Figure S2.6 The magnitude spectrum of $h(t)$. [Problem 2.29(e)]

a high-pass filter since it also filters out high frequency components. As $T \to 0$, the pass band of the system moves to higher frequencies, and the system tends toward a high-pass filter.

Solution 2.30

(a) The inverse Fourier transform of $\hat{H}(\varrho)$ is

$$\begin{aligned}
\hat{h}(r) &= \mathcal{F}^{-1}\{\hat{H}(\varrho)\} \\
&= \int_{-\infty}^{\infty} \hat{H}(\varrho) e^{j2\pi r\varrho} d\varrho \\
&= \int_{-\varrho_0}^{\varrho_0} |\varrho| e^{j2\pi r\varrho} d\varrho \\
&= \int_{0}^{\varrho_0} \varrho e^{j2\pi r\varrho} d\varrho - \int_{-\varrho_0}^{0} \varrho e^{j2\pi r\varrho} d\varrho \\
&= \int_{0}^{\varrho_0} \varrho e^{j2\pi r\varrho} d\varrho + \int_{0}^{-\varrho_0} \varrho e^{-j2\pi r\varrho} d\varrho \\
&= \int_{0}^{\varrho_0} \varrho \left[e^{j2\pi r\varrho} + e^{-j2\pi r\varrho} \right] d\varrho \\
&= 2\int_{0}^{\varrho_0} \varrho \cos(2\pi r\varrho) d\varrho \\
&= 2\left[\frac{\varrho \sin(2\pi r\varrho)}{2\pi r}\bigg|_{\varrho=0}^{\varrho_0} - \int_{0}^{\varrho_0} \frac{\sin(2\pi r\varrho)}{2\pi r} d\varrho \right] \\
&= 2\left[\frac{\varrho \sin(2\pi r\varrho_0)}{2\pi r} + \frac{\cos(2\pi r\varrho)}{4\pi^2 r^2}\bigg|_{\varrho=0}^{\varrho_0} \right] \\
&= \frac{1}{2\pi^2 r^2}\left[\cos(2\pi r\varrho_0) + 2\pi r\varrho_0 \sin(2\pi r\varrho_0) - 1 \right]
\end{aligned}$$

(b) The response of the filter is $g(r) = f(r) * \hat{h}(r)$, hence $G(\varrho) = F(\varrho)\hat{H}(\varrho)$. i) A constant function $f(r) = c$ has a Fourier transform of

$$F(\varrho) = c\delta(\varrho)$$

The transfer function of a ramp filter has a value zero at $\varrho = 0$. So the system response has a Fourier transform of

$$G(\varrho) = 0$$

Therefore, the responses of a ramp filter to a constant function is $g(r) = 0$. ii) Fourier transform of a sinusoid function $f(r) = \sin(\omega r)$ if

$$F(\varrho) = \frac{1}{2j}\left[\delta(\varrho - \frac{\omega}{2\pi}) - \delta(\varrho + \frac{\omega}{2\pi})\right]$$

Hence,

$$G(\varrho) = \begin{cases} \frac{\omega}{4\pi j}\left[\delta\left(\varrho - \frac{\omega}{2\pi}\right) - \delta\left(\varrho + \frac{\omega}{2\pi}\right)\right] & \varrho_0 = \omega \\ 0 & \text{otherwise} \end{cases}$$

Therefore, the response of a ramp filter to a sinusoid function is

$$g(r) = \begin{cases} \frac{\omega}{2\pi}\sin(\omega r) & \varrho_0 = \omega \\ 0 & \text{otherwise} \end{cases}$$

Solution 2.31 Suppose the Fourier transform of $f(x, y)$ is $F(u, v)$. Using the scaling properties, we have that the Fourier transform of $f(ax, by)$ is $\frac{1}{|ab|}F\left(\frac{u}{a}, \frac{v}{b}\right)$. The output of the system is $g(x, y)$:

$$\begin{aligned} g(x, y) &= \mathcal{F}\left\{\frac{1}{|ab|}F\left(\frac{u}{a}, \frac{v}{b}\right)\right\} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{|ab|}F\left(\frac{u}{a}, \frac{v}{b}\right)e^{-j2\pi(ux+vy)}dudv \\ &= \frac{1}{|ab|}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(\xi, \eta)e^{j2\pi(a\xi(-x)+b\eta(-y))}|ab|d\xi d\eta \end{aligned}$$

Note that

$$f(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(u, v)e^{j2\pi(ux+vy)}dudv$$

we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(\xi, \eta)e^{j2\pi(a\xi(-x)+b\eta(-y))}|ab|d\xi d\eta = |ab|f(-ax, -by).$$

Therefore, $g(x, y) = f(-ax, -by)$ is a scaled and inverted replica of the input.

Solution 2.32 The Fourier transform of the signal $f(x, y)$ and the noise $\eta(x, y)$ are:

$$\begin{aligned} F(u, v) &= \mathcal{F}\{f(x, y)\} \\ &= |ab|\mathcal{F}\{\text{sinc}(ax, by)\} \\ &= |ab|\left\{\frac{1}{|ab|}\text{rect}\left(\frac{u}{a}, \frac{v}{b}\right)\right\} \end{aligned}$$

$$
\begin{aligned}
&= \text{rect}\left(\frac{u}{a}, \frac{v}{b}\right) \\
&= \begin{cases} 1, & |x| < |a|/2 \text{ and } |y| < |b|/2 \\ 0, & \text{otherwise} \end{cases} \\
E(u, v) &= \mathcal{F}\{\eta(x, y)\} \\
&= \frac{1}{2}[\delta(u - A, v - B) + \delta(u + A, v + B)]
\end{aligned}
$$

Using the linearity of Fourier transform, the Fourier transform of the measurements $g(x, y)$ is:

$$G(u, v) = \text{rect}\left(\frac{u}{a}, \frac{v}{b}\right) + \frac{1}{2}[\delta(u - A, v - B) + \delta(u + A, v + B)]$$

which is plotted in Figure S2.7.

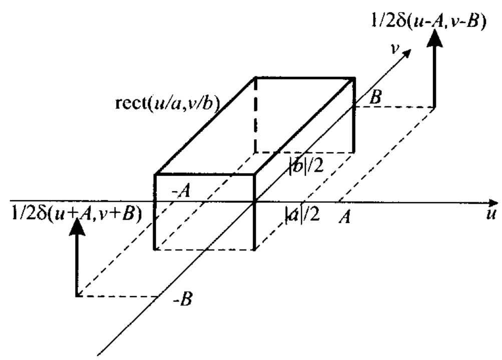

Figure S2.7 The Fourier transform of $g(x, y)$. [Problem 2.32]

In order for an ideal low pass filter to recover $f(x, y)$, the cutoff frequencies of the filter must satisfy

$$|a|/2 < U < A \text{ and } |b|/2 < V < B$$

The Fourier transform of $h(x, y)$ is $\text{rect}\left(\frac{u}{2U}, \frac{v}{2V}\right)$. The impulse response is

$$h(x, y) = \mathcal{F}^{-1}\left\{\text{rect}\left(\frac{u}{2U}, \frac{v}{2V}\right)\right\} = 4UV\,\text{sinc}(2Ux)\,\text{sinc}(2Vy).$$

For given a and b, we need $A > |a|/2$ and $B > |b|/2$. Otherwise we can not find an ideal low pass filter to exactly recover $f(x, y)$.

Solution 2.33

(a) The continuous Fourier transform of a rect function is a sinc function. Using the scaling property of the Fourier transform, we have:

$$G(u) = \mathcal{F}_{1D}\{g(x)\} = 2\,\text{sinc}(2u).$$

A sinc function, $\text{sinc}(x)$, is shown in Figure 2.4 (b).

(b) If the sampling period is $\Delta x_1 = 1/2$, we have

$$g_1(m) = g(m/2) = \begin{cases} 1, & -2 \le m \le 2 \\ 0, & \text{otherwise} \end{cases}.$$

Its DTFT is

$$\begin{aligned} G_1(\omega) &= \mathcal{F}_{\text{DTFT}}\{g_1(m)\} \\ &= e^{j2\omega} + e^{j\omega} + 1e^{j0\omega} + e^{-j\omega} + 2e^{-j2\omega} \\ &= 1 + 2\cos(\omega) + 2\cos(2\omega). \end{aligned}$$

The DTFT of $g_1(m)$ is shown in Figure S2.8.

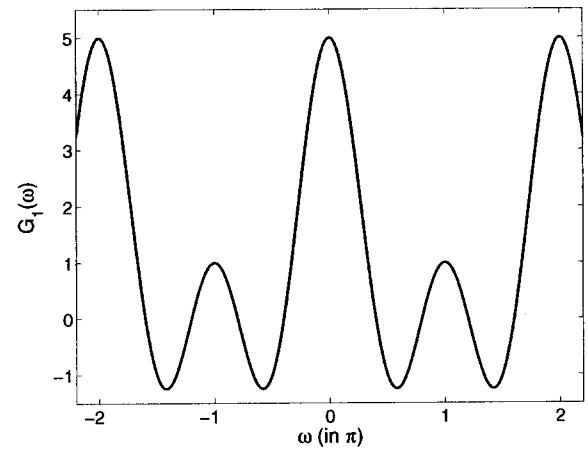

Figure S2.8 The discrete-time Fourier transform $g_1(m)$. [Problem 2.33(b)]

(c) If the sampling period is $\Delta x_2 = 1$, we have

$$g_2(m) = g(m) = \begin{cases} 1, & -1 \le m \le 1 \\ 0, & \text{otherwise} \end{cases}.$$

Its DTFT is

$$\begin{aligned} G_2(\omega) &= \mathcal{F}_{\text{DTFT}}\{g_2(m)\} \\ &= e^{j\omega} + 1e^{j0\omega} + e^{-j\omega} \\ &= 1 + 2\cos(\omega). \end{aligned}$$

The DTFT of $g_2(m)$ is shown in Figure S2.9.

(d) The discrete version of signal $g(x)$ can be written as:

$$g_1(m) = g(x - m\Delta x_1), \quad m = -\infty, \cdots, -1, 0, 1, \cdots, +\infty.$$

The DTFT of $g_1(m)$ is

$$G_1(\omega) = \mathcal{F}_{\text{DTFT}}\{g_1(m)\}$$

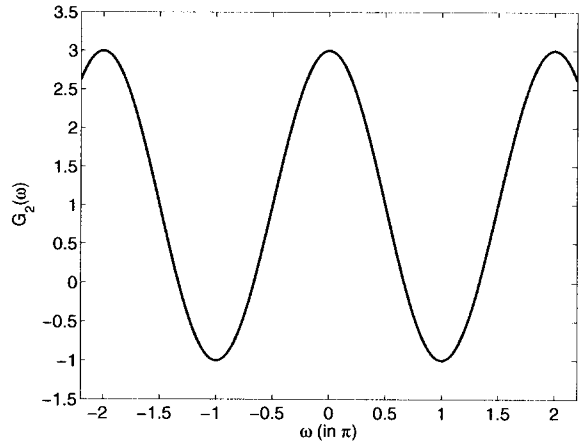

Figure S2.9 The discrete-time Fourier transform $g_2(m)$. [Problem 2.33(c)]

$$
\begin{aligned}
&= \sum_{m=-\infty}^{+\infty} g_1(m)e^{-j\omega m} \\
&= \sum_{m=-\infty}^{+\infty} g(m\Delta x_1)e^{-j\omega m} \\
&= \int_{-\infty}^{\infty} g(x)\delta_s(x;\,\Delta x_1)e^{-j\omega\frac{x}{\Delta x_1}}\,dx.
\end{aligned}
$$

In the above, $\delta_s(x;\,\Delta x_1)$ is the sampling function with the space between impulses equal to Δx_1. Because of the sampling function, we are able to convert the summation into integration. The last equation in the above is the continuous Fourier transform of the product of $g(x)$ and $\delta_s(x;\,\Delta x_1)$ evaluated as $u = \omega/(2\pi\Delta x_1)$. Using the product property of the continuous Fourier transform, we have:

$$
\begin{aligned}
G_1(\omega) &= \mathcal{F}\{g(x)\} * \mathcal{F}\{\delta_s(x;\,\Delta x_1)\}|_{u=\omega/(2\pi\Delta x_1)} \\
&= G(u) * \text{comb}(u\Delta x_1)|_{u=\omega/(2\pi\Delta x_1)}\,.
\end{aligned}
$$

The convolution of $G(u)$ and $\text{comb}(u\Delta x_1)$ is to replicate $G(u)$ to $u = k/\Delta x_1$. Since $u = \omega/(2\pi\Delta x_1)$, $G_1(\omega)$ is periodic with period $\Omega = 2\pi$.

(e) The proof is similar to that for the continuous Fourier transform:

$$
\begin{aligned}
\mathcal{F}_{\text{DTFT}}\{x(m) * y(m)\} &= \mathcal{F}_{\text{DTFT}}\{x(m) * y(m)\} \\
&= \mathcal{F}_{\text{DTFT}}\left\{\sum_{n=-\infty}^{\infty} x(m-n)y(n)\right\} \\
&= \sum_{m=-\infty}^{\infty} e^{-j\omega m} \sum_{n=-\infty}^{\infty} x(m-n)y(n)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n=-\infty}^{\infty}\left[\sum_{m=-\infty}^{\infty} e^{-j\omega m}x(m-n)\right]y(n) \\
&= \sum_{n=-\infty}^{\infty} e^{-j\omega n}\left[\sum_{k=-\infty}^{\infty} e^{-j\omega k}x(k)\right]y(n) \\
&\quad (\text{let } k = m-n) \\
&= \sum_{n=-\infty}^{\infty} e^{-j\omega n}\mathcal{F}_{\mathrm{DTFT}}\{x(m)\}y(n) \\
&= \mathcal{F}_{\mathrm{DTFT}}\{x(m)\}\mathcal{F}_{\mathrm{DTFT}}\{y(m)\}.
\end{aligned}
$$

(f) First we evaluate the convolution of $g_1(m)$ with $g_2(m)$:

$$
g_1(m) * g_2(m) = \begin{cases} 3, & -1 \le m \le 1 \\ 2, & m = \pm 2 \\ 1, & m = \pm 3 \\ 0, & \text{otherwise} \end{cases}.
$$

Then by direct computation, we have

$$
\begin{aligned}
\mathcal{F}_{\mathrm{DTFT}}\{g_1(m) * g_2(m)\} &= 3 + 3 \times 2\cos(\omega) + 2 \times 2\cos(2\omega) + 2\cos(3\omega) \\
&= 3 + 6\cos(\omega) + 4\cos(2\omega) + 2\cos(3\omega).
\end{aligned}
$$

On the other hand, we have

$$
\mathcal{F}_{\mathrm{DTFT}}\{g_1(m)\} = 1 + 2\cos(\omega) + 2\cos(2\omega)
$$

and

$$
\mathcal{F}_{\mathrm{DTFT}}\{g_2(m)\} = 1 + 2\cos(\omega).
$$

So, the product of the DTFT's of $g_1(m)$ and $g_2(m)$ is

$$
\begin{aligned}
\mathcal{F}_{\mathrm{DTFT}}\{g_1(m)\}\mathcal{F}_{\mathrm{DTFT}}\{g_2(m)\} &= [1 + 2\cos(\omega)][1 + 2\cos(\omega) + 2\cos(2\omega)] \\
&= 1 + 4\cos(\omega) + 2\cos(2\omega) \\
&\quad +4\cos^2(\omega) + 4\cos(\omega)\cos(2\omega) \\
&= 1 + 4\cos(\omega) + 2\cos(2\omega) \\
&\quad +4\frac{1+\cos(2\omega)}{2} + 4\frac{\cos(\omega)+\cos(3\omega)}{2} \\
&= 3 + 6\cos(\omega) + 4\cos(2\omega) + 2\cos(3\omega).
\end{aligned}
$$

Therefore,

$$
\mathcal{F}_{\mathrm{DTFT}}\{g_1(m) * g_2(m)\} = \mathcal{F}_{\mathrm{DTFT}}\{g_1(m)\}\mathcal{F}_{\mathrm{DTFT}}\{g_2(m)\}.
$$