C H A P T E R 1
Systems of Linear Equations

CHAPTER 1
Systems of Linear Equations

Section 1.1 Introduction to Systems of Linear Equations

2. Because the term xy cannot be rewritten as $ax + by$ for any real numbers a and b, the equation cannot be written in the form $a_1x + a_2y = b$. So this equation is *not* linear in the variables x and y.

4. Because the terms x^2 and y^2 cannot be rewritten as $ax + by$ for any real numbers a and b, the equation cannot be written in the form $a_1x + a_2y = b$. So this equation is *not* linear in the variables x and y.

6. Because the equation is in the form $a_1x + a_2y = b$, it *is* linear in the variables x and y.

8. Choosing y as the free variable, let $y = t$ and obtain

$$3x - \tfrac{1}{2}t = 9$$
$$3x = 9 + \tfrac{1}{2}t$$
$$x = 3 + \tfrac{1}{6}t.$$

So, you can describe the solution set as $x = 3 + \frac{1}{6}t$ and $y = t$, where t is any real number.

10. Choosing x_2 and x_3 as free variables, let $x_3 = t$ and $x_2 = s$ and obtain $13x_1 - 26x + 39t = 13$.

Dividing this equation by 13 you obtain

$$x_1 - 2s + 3t = 1$$
$$x_1 = 1 + 2s - 3t.$$

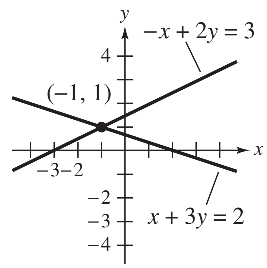
So, you can describe the solution set as $x_1 = 1 + 2s - 3t$, $x_2 = s$, and $x_3 = t$, where t and s are any real numbers.

12. From Equation 2 you have $x_2 = 3$. Substituting this value into Equation 1 produces $2x_1 - 12 = 6$ or $x_1 = 9$. So, the system has exactly one solution: $x_1 = 9$ and $x_2 = 3$.

14. From Equation 3 you conclude that $z = 2$. Substituting this value into Equation 2 produces $2y + 2 = 6$ or $y = 2$. Finally, substituting $y = 2$ and $z = 2$ into Equation 1, you obtain $x - 2 = 4$ or $x = 6$. So, the system has exactly one solution: $x = 6$, $y = 2$, and $z = 2$.

16. From the second equation you have $x_2 = 0$. Substituting this value into Equation 1 produces $x_1 + x_3 = 0$. Choosing x_3 as the free variable, you have $x_3 = t$ and obtain $x_1 + t = 0$ or $x_1 = -t$. So, you can describe the solution set as $x_1 = -t$, $x_2 = 0$, and $x_3 = t$.

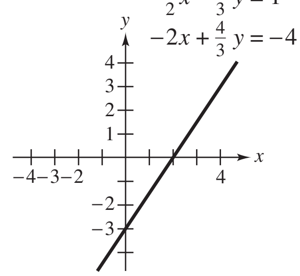
18.



$$x + 3y = 2$$
$$-x + 2y = 3$$

Adding the first equation to the second equation produces a new second equation, $5y = 5$, or $y = 1$. So, $x = 2 - 3y = 2 - 3(1)$, and the solution is: $x = -1$, $y = 1$. This is the point where the two lines intersect.

20.



The two lines coincide.

Multiplying the first equation by 2 produces a new first equation.

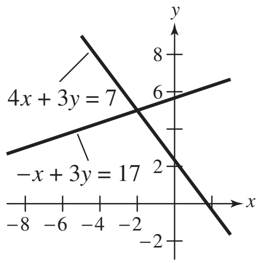
$$x - \tfrac{2}{3}y = 2$$
$$-2x + \tfrac{4}{3}y = -4$$

Adding 2 times the first equation to the second equation produces a new second equation.

$$x - \tfrac{2}{3}y = 2$$
$$0 = 0$$

Choosing $y = t$ as the free variable, you obtain $x = \frac{2}{3}t + 2$. So, you can describe the solution set as $x = \frac{2}{3}t + 2$ and $y = t$, where t is any real number.

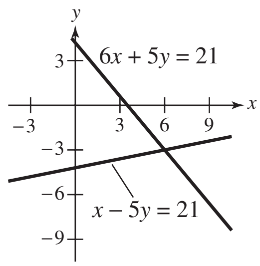
22.



$$-x + 3y = 17$$
$$4x + 3y = 7$$

Subtracting the first equation from the second equation produces a new second equation, $5x = -10$, or $x = -2$. So, $4(-2) + 3y = 7$, or $y = 5$, and the solution is: $x = -2$, $y = 5$. This is the point where the two lines intersect.

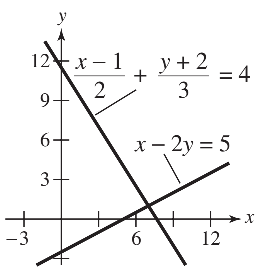
24.



$$x - 5y = 21$$
$$6x + 5y = 21$$

Adding the first equation to the second equation produces a new second equation, $7x = 42$, or $x = 6$. So, $6 - 5y = 21$, or $y = -3$, and the solution is: $x = 6$, $y = -3$. This is the point where the two lines intersect.

26.



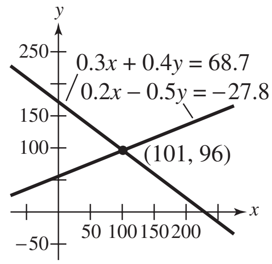
$$\frac{x-1}{2} + \frac{y+2}{3} = 4$$
$$x - 2y = 5$$

Multiplying the first equation by 6 produces a new first equation.

$$3x + 2y = 23$$
$$x - 2y = 5$$

Adding the first equation to the second equation produces a new second equation, $4x = 28$, or $x = 7$. So, $7 - 2y = 5$, or $y = 1$, and the solution is: $x = 7$, $y = 1$. This is the point where the two lines intersect.

28.



$$0.2x - 0.5y = -27.8$$
$$0.3x + 0.4y = 68.7$$

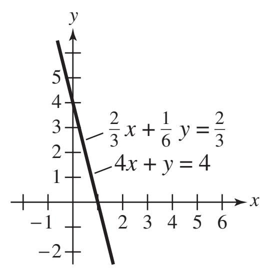
Multiplying the first equation by 40 and the second equation by 50 produces new equations.

$$8x - 20y = -1112$$
$$15x + 20y = 3435$$

Adding the first equation to the second equation produces a new second equation, $23x = 2323$, or $x = 101$.

So, $8(101) - 20y = -1112$, or $y = 96$, and the solution is: $x = 101$, $y = 96$. This is the point where the two lines intersect.

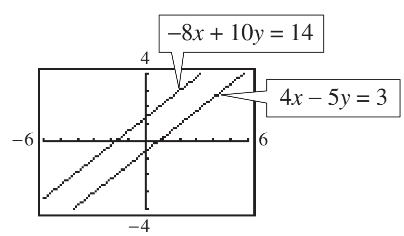
30.



$$\tfrac{2}{3}x + \tfrac{1}{6}y = \tfrac{2}{3}$$
$$4x + y = 4$$

Adding 6 times the first equation to the second equation produces a new second equation, $0 = 0$. Choosing $x = t$ as the free variable, you obtain $y = 4 - 4t$. So, you can describe the solution as $x = t$ and $y = 4 - 4t$, where t is any real number.

32. (a)



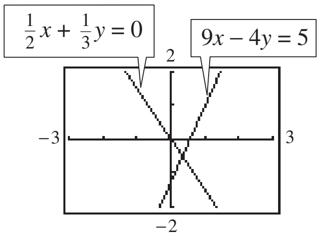
(b) This system is inconsistent, because you see two parallel lines on the graph of the system.

(c) Because the system is inconsistent, you cannot approximate the solution.

(d) Adding -2 times the first equation to the second you obtain $0 = 8$, which is a false statement. This system has no solution.

(e) You obtained the same answer both geometrically and algebraically.

34. (a)



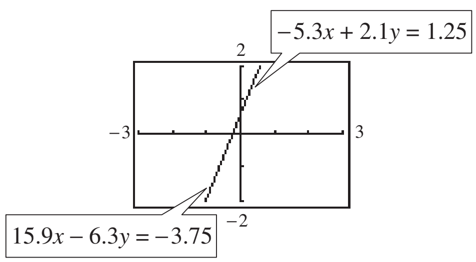
(b) Two lines corresponding to two equations intersect at a point, so this system has a unique solution.

(c) $x \approx \frac{1}{3},\ y \approx -\frac{1}{2}$

(d) Adding −18 times the second equation to the first equation you obtain $-10y = 5$ or $y = -\frac{1}{2}$. Substituting $y = -\frac{1}{2}$ into the first equation you obtain $9x = 3$ or $x = \frac{1}{3}$. The solution is: $x = \frac{1}{3}$ and $y = -\frac{1}{2}$.

(e) You obtained the same answer both geometrically and algebraically.

36. (a)



(b) Because each equation has the same line as a graph, there are infinitely many solutions.

(c) All solutions of this system lie on the line $y = \frac{53}{21}x + \frac{25}{42}$. So let $x = t$, then the solution set is $x = t,\ y = \frac{53}{21}t + \frac{25}{42}$ for any real number t.

(d) Adding 3 times the first equation to the second equation you obtain

$$\begin{aligned} -5.3x + 2.1y &= 1.25 \\ 0 &= 0. \end{aligned}$$

Choosing $x = t$ as the free variable, you obtain $2.1y = 5.3t + 1.25$ or $21y = 53t + 12.5$ or $y = \frac{53}{21}t + \frac{25}{42}$. So, you can describe the solution set as $x = t,\ y = \frac{53}{21}t + \frac{25}{42}$ for any real number t.

(e) You obtained the same answer both geometrically and algebraically.

38. Adding −2 times the first equation to the second equation produces a new second equation.

$$\begin{aligned} 3x + 2y &= 2 \\ 0 &= 10 \end{aligned}$$

Because the second equation is a false statement, the original system of equations has no solution.

40. Adding −6 times the first equation to the second equation produces a new second equation.

$$\begin{aligned} x_1 - 2x_2 &= 0 \\ 14x_2 &= 0 \end{aligned}$$

Now, using back-substitution, the system has exactly one solution: $x_1 = 0$ and $x_2 = 0$.

42. Multiplying the first equation by $\frac{3}{2}$ produces a new first equation.

$$\begin{aligned} x_1 + \tfrac{1}{4}x_2 &= 0 \\ 4x_1 + x_2 &= 0 \end{aligned}$$

Adding −4 times the first equation to the second equation produces a new second equation.

$$\begin{aligned} x_1 + \tfrac{1}{4}x_2 &= 0 \\ 0 &= 0 \end{aligned}$$

Choosing $x_2 = t$ as the free variable, you obtain $x_1 = -\frac{1}{4}t$. So you can describe the solution set as $x_1 = -\frac{1}{4}t$ and $x_2 = t$, where t is any real number.

44. To begin, change the form of the first equation.

$$\begin{aligned} \frac{x_1}{4} + \frac{x_2}{3} &= \frac{7}{12} \\ 2x_1 - x_2 &= 12 \end{aligned}$$

Multiplying the first equation by 4 yields a new first equation.

$$\begin{aligned} x_1 + \tfrac{4}{3}x_2 &= \tfrac{7}{3} \\ 2x_1 - x_2 &= 12 \end{aligned}$$

Adding −2 times the first equation to the second equation produces a new second equation.

$$\begin{aligned} x_1 + \tfrac{4}{3}x_2 &= \tfrac{7}{3} \\ -\tfrac{11}{3}x_2 &= \tfrac{22}{3} \end{aligned}$$

Multiplying the second equation by $-\frac{3}{11}$ yields a new second equation.

$$\begin{aligned} x_1 + \tfrac{4}{3}x_2 &= \tfrac{7}{3} \\ x_2 &= -2 \end{aligned}$$

Now, using back-substitution, the system has exactly one solution: $x_1 = 5$ and $x_2 = -2$.

46. Multiplying the first equation by 20 and the second equation by 100 produces a new system.

$$\begin{aligned} x_1 - 0.6x_2 &= 4.2 \\ 7x_1 + 2x_2 &= 17 \end{aligned}$$

Adding −7 times the first equation to the second equation produces a new second equation.

$$\begin{aligned} x_1 - 0.6x_2 &= 4.2 \\ 6.2x_2 &= -12.4 \end{aligned}$$

Now, using back-substitution, the system has exactly one solution: $x_1 = 3$ and $x_2 = -2$.

48. Adding the first equation to the second equation yields a new second equation.

$$\begin{aligned} x + y + z &= 2 \\ 4y + 3z &= 10 \\ 4x + y &= 4 \end{aligned}$$

Adding -4 times the first equation to the third equation yields a new third equation.

$$\begin{aligned} x + y + z &= 2 \\ 4y + 3z &= 10 \\ -3y - 4z &= -4 \end{aligned}$$

Dividing the second equation by 4 yields a new second equation.

$$\begin{aligned} x + y + z &= 2 \\ y + \tfrac{3}{4}z &= \tfrac{5}{2} \\ -3y - 4z &= -4 \end{aligned}$$

Adding 3 times the second equation to the third equation yields a new third equation.

$$\begin{aligned} x + y + z &= 2 \\ y + \tfrac{3}{4}z &= \tfrac{5}{2} \\ -\tfrac{7}{4}z &= \tfrac{7}{2} \end{aligned}$$

Multiplying the third equation by $-\frac{4}{7}$ yields a new third equation.

$$\begin{aligned} x + y + z &= 2 \\ y + \tfrac{3}{4}z &= \tfrac{5}{2} \\ z &= -2 \end{aligned}$$

Now, using back-substitution, the system has exactly one solution: $x = 0$, $y = 4$, and $z = -2$.

50. Interchanging the first and third equations yields a new system.

$$\begin{aligned} x_1 - 11x_2 + 4x_3 &= 3 \\ 2x_1 + 4x_2 - x_3 &= 7 \\ 5x_1 - 3x_2 + 2x_3 &= 3 \end{aligned}$$

Adding -2 times the first equation to the second equation yields a new second equation.

$$\begin{aligned} x_1 - 11x_2 + 4x_3 &= 3 \\ 26x_2 - 9x_3 &= 1 \\ 5x_1 - 3x_2 + 2x_3 &= 3 \end{aligned}$$

Adding -5 times the first equation to the third equation yields a new third equation.

$$\begin{aligned} x_1 - 11x_2 + 4x_3 &= 3 \\ 26x_2 - 9x_3 &= 1 \\ 52x_2 - 18x_3 &= -12 \end{aligned}$$

At this point you realize that Equations 2 and 3 cannot both be satisfied. So, the original system of equations has no solution.

52. Adding -4 times the first equation to the second equation and adding -2 times the first equation to the third equation produces new second and third equations.

$$\begin{aligned} x_1 + 4x_3 &= 13 \\ -2x_2 - 15x_3 &= -45 \\ -2x_2 - 15x_3 &= -45 \end{aligned}$$

The third equation can be disregarded because it is the same as the second one. Choosing x_3 as a free variable and letting $x_3 = t$, you can describe the solution as

$x_1 = 13 - 4t$

$x_2 = \frac{45}{2} - \frac{15}{2}t$

$x_3 = t$, where t is any real number.

54. Adding -3 times the first equation to the second equation produces a new second equation.

$$\begin{aligned} x_1 - 2x_2 + 5x_3 &= 2 \\ 8x_2 - 16x_3 &= -8 \end{aligned}$$

Dividing the second equation by 8 yields

$$\begin{aligned} x_1 - 2x_2 + 5x_3 &= 2 \\ x_2 - 2x_3 &= -1. \end{aligned}$$

Adding 2 times the second equation to the first equation yields

$$\begin{aligned} x_1 + x_3 &= 0 \\ x_2 - 2x_3 &= -1. \end{aligned}$$

Letting $x_3 = t$ be the free variable, you can describe the solution as $x_1 = -t$, $x_2 = 2t - 1$, and $x_3 = t$, where t is any real number.

56. Adding -2 times the first equation to the fourth equation, yields

$$\begin{aligned} x_1 \quad\quad\quad + 3x_4 &= 4 \\ 2x_2 - x_3 - x_4 &= 0 \\ 3x_2 \quad - 2x_4 &= 1 \\ -x_2 + 4x_3 - 6x_4 &= -3. \end{aligned}$$

Multiplying the fourth equation by -1, and interchanging it with the second equation, yields

$$\begin{aligned} x_1 \quad\quad\quad + 3x_4 &= 4 \\ x_2 - 4x_3 + 6x_4 &= 3 \\ 3x_2 \quad - 2x_4 &= 1 \\ 2x_2 - x_3 - x_4 &= 0. \end{aligned}$$

Adding -3 times the second equation to the third, and -2 times the second equation to the fourth, produces

$$\begin{aligned} x_1 \quad\quad\quad + 3x_4 &= 4 \\ x_2 - 4x_3 + 6x_4 &= 3 \\ 12x_3 - 20x_4 &= -8 \\ 7x_3 - 13x_4 &= -6. \end{aligned}$$

Dividing the third equation by 12 yields

$$\begin{aligned} x_1 \quad\quad\quad + 3x_4 &= 4 \\ x_2 - 4x_3 + 6x_4 &= 3 \\ x_3 - \tfrac{5}{3}x_4 &= -\tfrac{2}{3} \\ 7x_3 - 13x_4 &= -6. \end{aligned}$$

Adding -7 times the third equation to the fourth yields

$$\begin{aligned} x_1 \quad\quad\quad + 3x_4 &= 4 \\ x_2 - 4x_3 + 6x_4 &= 3 \\ x_3 - \tfrac{5}{3}x_4 &= -\tfrac{2}{3} \\ \tfrac{4}{3}x_4 &= \tfrac{4}{3}. \end{aligned}$$

Using back-substitution, the original system has exactly one solution: $x_1 = 1$, $x_2 = 1$, $x_3 = 1$ and $x_4 = 1$.

Answers may vary slightly for Exercises 58–64.

58. Using a computer software program or graphing utility, you obtain $x = 10$, $y = -20$, $z = 40$, $w = -12$.

60. Using a computer software program or graphing utility, you obtain $x = 0.8$, $y = 1.2$, $z = -2.4$.

62. Using a computer software program or graphing utility, you obtain $x_1 = 0.6$, $x_2 = -0.5$, $x_3 = 0.8$.

64. Using a computer software program or graphing utility, you obtain $x = 6.8813$, $y = 163.3111$, $z = 210.2915$, $w = 59.2913$.

66. $x = y = z = 0$ is clearly a solution.

Dividing the first equation by 2 produces

$$\begin{aligned} x + \tfrac{3}{2}y \quad\quad &= 0 \\ 4x + 3y - z &= 0 \\ 8x + 3y + 3z &= 0. \end{aligned}$$

Adding -4 times the first equation to the second equation, and -8 times the first equation to the third, yields

$$\begin{aligned} x + \tfrac{3}{2}y \quad\quad &= 0 \\ -3y - z &= 0 \\ -9y + 3z &= 0. \end{aligned}$$

Adding -3 times the second equation to the third equation yields

$$\begin{aligned} x + \tfrac{3}{2}y \quad\quad &= 0 \\ -3y - z &= 0 \\ 6z &= 0. \end{aligned}$$

Using back-substitution you conclude there is exactly one solution: $x = y = z = 0$.

68. $x = y = z = 0$ is clearly a solution.

Dividing the first equation by 12 yields

$$\begin{aligned} x + \tfrac{5}{12}y + \tfrac{1}{12}z &= 0 \\ 12x + 4y - z &= 0. \end{aligned}$$

Adding -12 times the first equation one to the second yields

$$\begin{aligned} x + \tfrac{5}{12}y + \tfrac{1}{12}z &= 0 \\ -y - 2z &= 0. \end{aligned}$$

Letting $z = t$ be the free variable, you can describe the solution as $x = \frac{3}{4}t$, $y = -2t$, $z = t$, where t is any real number.

70. (a) False. Any system of linear equations is either consistent which means it has a unique solution, or infinitely many solutions; or inconsistent, i.e., it has no solution. This result is stated on page 6 of the text, and will be proved later in Theorem 2.5.

(b) True. See definition on page 7 of the text.

(c) False. Consider the following system of three linear equations with two variables.

$$\begin{aligned} 2x + y &= -3 \\ -6x - 3y &= 9 \\ x \quad\quad &= 1. \end{aligned}$$

The solution to this system is: $x = 1$, $y = -5$.

72. Because $x_1 = t$ and $x_2 = s$, you can write $x_3 = 3 + s - t = 3 + x_2 - x_1$. One system could be

$$\begin{aligned} x_1 - x_2 + x_3 &= 3 \\ -x_1 + x_2 - x_3 &= -3 \end{aligned}$$

Letting $x_3 = t$ and $x_2 = s$ be the free variables, you can describe the solution as $x_1 = 3 + s - t$, $x_2 = s$, $x_3 = t$, where t and s are any real numbers.

74. Substituting $A = \frac{1}{x}$ and $B = \frac{1}{y}$ into the original system yields

$$\begin{aligned} 2A + 3B &= 0 \\ 3A - 4B &= -\frac{25}{6}. \end{aligned}$$

Reduce the system to row-echelon form.

$$\begin{aligned} 8A + 12B &= 0 \\ 9A - 12B &= -\frac{25}{2} \end{aligned}$$

$$\begin{aligned} 8A + 12B &= 0 \\ 17A \qquad &= -\frac{25}{2} \end{aligned}$$

So, $A = -\frac{25}{34}$ and $B = \frac{25}{51}$. Because $A = \frac{1}{x}$ and $B = \frac{1}{y}$, the solution of the original system of equations is: $x = -\frac{34}{25}$ and $y = \frac{51}{25}$.

76. Substituting $A = \frac{1}{x}$, $B = \frac{1}{y}$, and $C = \frac{1}{z}$ into the original system yields

$$\begin{aligned} 2A + B - 2C &= 5 \\ 3A - 4B \qquad &= -1 \\ 2A + B + 3C &= 0. \end{aligned}$$

Reduce the system to row-echelon form.

$$\begin{aligned} 2A + B - 2C &= 5 \\ 3A - 4B \qquad &= -1 \\ 5C &= -5 \end{aligned}$$

$$\begin{aligned} 3A - 4B \qquad &= -1 \\ -11B + 6C &= -17 \\ 5C &= -5 \end{aligned}$$

So, $C = -1$. Using back-substitution, $-11B + 6(-1) = -17$, or $B = 1$ and $3A - 4(1) = -1$, or $A = 1$. Because $A = 1/x$, $B = 1/y$, and $C = 1/z$, the solution of the original system of equations is: $x = 1$, $y = 1$, and $z = -1$.

78. Multiplying the first equation by $\sin\theta$ and the second by $\cos\theta$ produces

$$\begin{aligned} (\sin\theta\cos\theta)x + (\sin^2\theta)y &= \sin\theta \\ -(\sin\theta\cos\theta)x + (\cos^2\theta)y &= \cos\theta \end{aligned}$$

Adding these two equations yields

$$\begin{aligned} (\sin^2\theta + \cos^2\theta)y &= \sin\theta + \cos\theta \\ y &= \sin\theta + \cos\theta. \end{aligned}$$

So,

$$(\cos\theta)x + (\sin\theta)y = (\cos\theta)x + \sin\theta(\sin\theta + \cos\theta) = 1 \text{ and}$$

$$x = \frac{(1 - \sin^2\theta - \sin\theta\cos\theta)}{\cos\theta} = \frac{(\cos^2\theta - \sin\theta\cos\theta)}{\cos\theta} = \cos\theta - \sin\theta.$$

Finally, the solution is $x = \cos\theta - \sin\theta$ and $y = \cos\theta + \sin\theta$.

80. Interchange the two equations and row reduce.

$$\begin{aligned} x - \tfrac{3}{2}y &= -6 \\ kx + y &= 4 \end{aligned}$$

$$\begin{aligned} x - \tfrac{3}{2}y &= -6 \\ \left(\tfrac{3}{2}k + 1\right)y &= 4 + 6k \end{aligned}$$

So, if $k = -\frac{2}{3}$, there will be an infinite number of solutions.

82. Reduce the system.

$$\begin{aligned} x + ky &= 2 \\ (1 - k^2)y &= 4 - 2k \end{aligned}$$

If $k = \pm 1$, there will be no solution.

84. Interchange the first two equations and row reduce

$$\begin{aligned} x + y + z &= 0 \\ ky + 2kz &= 4k \\ -3y - z &= 1 \end{aligned}$$

If $k = 0$, then there is an infinite number of solutions. Otherwise,

$$\begin{aligned} x + y + z &= 0 \\ y + 2z &= 4 \\ 5z &= 13. \end{aligned}$$

Because this system has exactly one solution, the answer is all $k \neq 0$.

86. Reducing the system to row-echelon form produces

$$\begin{aligned} x + \quad 5y + \quad z &= 0 \\ y - \quad 2z &= 0 \\ (a-10)y + (b-2)z &= c \end{aligned}$$

$$\begin{aligned} x + 5y + z &= 0 \\ y - 2z &= 0 \\ (2a + b - 22)z &= c \end{aligned}$$

So, you see that

(a) if $2a + b - 22 \neq 0$, then there is exactly one solution.

(b) if $2a + b - 22 = 0$ and $c = 0$, then there is an infinite number of solutions.

(c) if $2a + b - 22 = 0$ and $c \neq 0$, there is no solution.

88. If $c_1 = c_2 = c_3 = 0$, then the system is consistent because $x = y = 0$ is a solution.

90. Multiplying the first equation by c, and the second by a, produces

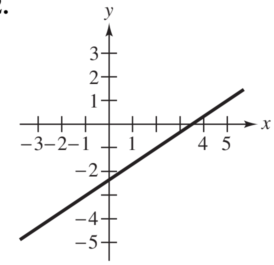
$$\begin{aligned} acx + bcy &= ec \\ acx + day &= af. \end{aligned}$$

Subtracting the second equation from the first yields

$$\begin{aligned} acx + bcy &= ec \\ (ad - bc)y &= af - ec. \end{aligned}$$

So, there is a unique solution if $ad - bc \neq 0$.

92.



The two lines coincide.

$$\begin{aligned} 2x - 3y &= 7 \\ 0 &= 0 \end{aligned}$$

Letting $y = t$, $x = \frac{7 + 3t}{2}$.

The graph does not change.

94.

$$\begin{aligned} 21x - 20y &= 0 \\ 13x - 12y &= 120 \end{aligned}$$

Subtracting 5 times the second equation from 3 times the first equation produces a new first equation, $-2x = -600$, or $x = 300$. So, $21(300) - 20y = 0$, or $y = 315$, and the solution is: $x = 300$, $y = 315$. The graphs are misleading because they appear to be parallel, but they actually intersect at $(300, 315)$.

Section 1.2 Gaussian Elimination and Gauss-Jordan Elimination

2. Because the matrix has 2 rows and 4 columns, it has size 2×4.

4. Because the matrix has 1 row and 5 columns, it has size 1×5.

6. Because the matrix has 1 row and 1 column, it has size 1×1.

8. Because the matrix has 4 rows and 1 column, it has size 4×1.

10. Because the leading 1 in the first row is not farther to the left than the leading 1 in the second row, the matrix is not in row-echelon form.

12. The matrix satisfies all three conditions in the definition of row-echelon form. However, because the third column does not have zeros above the leading 1 in the third row, the matrix is not in reduced row-echelon form.

14. The matrix satisfies all three conditions in the definition of row-echelon form. Moreover, because each column that has a leading 1 (columns one and four) has zeros elsewhere, the matrix is in reduced row-echelon form.

16. Because the matrix is in reduced row-echelon form, you can convert back to a system of linear equations

$$\begin{aligned} x_1 &= 2 \\ x_2 &= 3. \end{aligned}$$

18. Because the matrix is in row-echelon form, you can convert back to a system of linear equations

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ x_3 &= -1. \end{aligned}$$

Using back-substitution, you have $x_3 = -1$. Letting $x_2 = t$ be the free variable, you can describe the solution as $x_1 = 1 - 2t$, $x_2 = t$ and $x_3 = -1$, where t is any real number.

20. Gaussian elimination produces the following.

$$\begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & -2 & 1 & -2 \\ 1 & 0 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & -2 & 1 & -2 \\ 2 & 1 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & -2 \\ 2 & 1 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Because the matrix is in row-echelon form, convert back to a system of linear equations.

$$\begin{aligned} x_1 \qquad + x_3 &= 0 \\ x_2 \qquad &= 1 \\ x_3 &= 1 \end{aligned}$$

By back-substitution $x_1 = -x_3 = -1$. So, the solution is: $x_1 = -1$, $x_2 = 1$, and $x_3 = 1$.

22. Because the fourth row of this matrix corresponds to the equation $0 = 2$, there is no solution to the linear system.

24. The augmented matrix for this system is

$$\begin{bmatrix} 2 & 6 & 16 \\ -2 & -6 & -16 \end{bmatrix}.$$

Use Gauss-Jordan elimination as follows.

$$\begin{bmatrix} 2 & 6 & 16 \\ -2 & -6 & -16 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 8 \\ -2 & -6 & -16 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

Converting back to system of linear equations, you have

$x + 3y = 8$.

Choosing $y = t$ as the free variable, you can describe the solution as $x = 8 - 3t$ and $y = t$, where t is any real number.

26. The augmented matrix for this system is

$$\begin{bmatrix} 2 & -1 & -0.1 \\ 3 & 2 & 1.6 \end{bmatrix}.$$

Gaussian elimination produces the following.

$$\begin{bmatrix} 2 & -1 & -0.1 \\ 3 & 2 & 1.6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{20} \\ 3 & 2 & \frac{8}{5} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{20} \\ 0 & \frac{7}{2} & \frac{7}{4} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{20} \\ 0 & 1 & \frac{1}{2} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & \frac{1}{5} \\ 0 & 1 & \frac{1}{2} \end{bmatrix}$$

Converting back to a system of equations, the solution is: $x = \frac{1}{5}$ and $y = \frac{1}{2}$.

28. The augmented matrix for this system is

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 6 \\ 3 & -2 & 8 \end{bmatrix}.$$

Gaussian elimination produces the following.

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 6 \\ 3 & -2 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 6 \\ 0 & -8 & 8 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -6 \\ 0 & -8 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -6 \\ 0 & 0 & -40 \end{bmatrix}$$

Because the third row corresponds to the equation $0 = -40$, you conclude that the system has no solution.

30. The augmented matrix for this system is

$$\begin{bmatrix} 2 & -1 & 3 & 24 \\ 0 & 2 & -1 & 14 \\ 7 & -5 & 0 & 6 \end{bmatrix}.$$

Gaussian elimination produces the following.

$$\begin{bmatrix} 2 & -1 & 3 & 24 \\ 0 & 2 & -1 & 14 \\ 7 & -5 & 0 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & 12 \\ 0 & 2 & -1 & 14 \\ 7 & -5 & 0 & 6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & 12 \\ 0 & 2 & -1 & 14 \\ 0 & -\frac{3}{2} & -\frac{21}{2} & -78 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & 12 \\ 0 & 1 & -\frac{1}{2} & 7 \\ 0 & 0 & -\frac{45}{4} & -\frac{135}{2} \end{bmatrix}$$

Back-substitution now yields

$x_3 = 6$

$x_2 = 7 + \frac{1}{2}x_3 = 7 + \frac{1}{2}(6) = 10$

$x_1 = 12 - \frac{3}{2}x_3 + \frac{1}{2}x_2 = 12 - \frac{3}{2}(6) + \frac{1}{2}(10) = 8.$

So, the solution is: $x_1 = 8$, $x_2 = 10$ and $x_3 = 6$.

32. The augmented matrix for this system is

$$\begin{bmatrix} 2 & 0 & 3 & 3 \\ 4 & -3 & 7 & 5 \\ 8 & -9 & 15 & 10 \end{bmatrix}.$$

Gaussian elimination produces the following.

$$\begin{bmatrix} 2 & 0 & 3 & 3 \\ 4 & -3 & 7 & 5 \\ 8 & -9 & 15 & 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{3}{2} \\ 4 & -3 & 7 & 5 \\ 8 & -9 & 15 & 10 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{3}{2} \\ 0 & -3 & 1 & -1 \\ 0 & -9 & 3 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{3}{2} \\ 0 & 1 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Because the third row corresponds to the equation $0 = 1$, there is no solution to the original system.

34. The augmented matrix for this system is

$$\begin{bmatrix} 1 & 2 & 1 & 8 \\ -3 & -6 & -3 & -21 \end{bmatrix}.$$

Gaussian elimination produces the following matrix.

$$\begin{bmatrix} 1 & 2 & 1 & 8 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Because the second row corresponds to the equation $0 = 3$, there is no solution to the original system.

36. The augmented matrix for this system is

$$\begin{bmatrix} 2 & 1 & -1 & 2 & -6 \\ 3 & 4 & 0 & 1 & 1 \\ 1 & 5 & 2 & 6 & -3 \\ 5 & 2 & -1 & -1 & 3 \end{bmatrix}$$

Gaussian elimination produces the following.

$$\begin{bmatrix} 1 & 5 & 2 & 6 & -3 \\ 3 & 4 & 0 & 1 & 1 \\ 2 & 1 & -1 & 2 & -6 \\ 5 & 2 & -1 & -1 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 5 & 2 & 6 & -3 \\ 0 & -11 & -6 & -17 & 10 \\ 0 & -9 & -5 & -10 & 0 \\ 0 & -23 & -11 & -31 & 18 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 5 & 2 & 6 & -3 \\ 0 & 1 & \frac{6}{11} & \frac{17}{11} & -\frac{10}{11} \\ 0 & -9 & -5 & -10 & 0 \\ 0 & -23 & -11 & -31 & 18 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 5 & 2 & 6 & -3 \\ 0 & 1 & \frac{6}{11} & \frac{17}{11} & -\frac{10}{11} \\ 0 & 0 & -\frac{1}{11} & \frac{43}{11} & -\frac{90}{11} \\ 0 & 0 & \frac{17}{11} & \frac{50}{11} & -\frac{32}{11} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 5 & 2 & 6 & -3 \\ 0 & 1 & \frac{6}{11} & \frac{17}{11} & -\frac{10}{11} \\ 0 & 0 & 1 & -43 & 90 \\ 0 & 0 & \frac{17}{11} & \frac{50}{11} & -\frac{32}{4} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 5 & 2 & 6 & -3 \\ 0 & 1 & \frac{6}{11} & \frac{17}{11} & -\frac{10}{11} \\ 0 & 0 & 1 & -43 & 90 \\ 0 & 0 & 0 & \frac{781}{11} & -\frac{1562}{11} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 5 & 2 & 6 & -3 \\ 0 & 1 & \frac{6}{11} & \frac{17}{11} & -\frac{10}{11} \\ 0 & 0 & 1 & -43 & 90 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

Back-substitution now yields

$w = -2$

$z = 90 + 43w = 90 + 43(-2) = 4$

$y = -\frac{10}{11} - \frac{6}{11}(z) - \frac{17}{11}(w) = -\frac{10}{11} - \frac{6}{11}(4) - \frac{17}{11}(-2) = 0$

$x = -3 - 5y - 2z - 6w = -3 - 5(0) - 2(4) - 6(-2) = 1.$

So, the solution is: $x = 1$, $y = 0$, $z = 4$ and $w = -2$.

38. Using a computer software program or graphing utility, you obtain

$x = 14.3629$

$y = 32.7569$

$z = 28.6356.$

40. Using a computer software program or graphing utility, you obtain

$x_1 = 2$

$x_2 = -1$

$x_3 = 3$

$x_4 = 4$

$x_5 = 1.$

42. Using a computer software program or graphing utility, you obtain

$x_1 = 1$

$x_2 = -1$

$x_3 = 2$

$x_4 = 0$

$x_5 = -2$

$x_6 = 1.$

44. The corresponding equations are

$$\begin{aligned} x_1 \qquad\quad &= 0 \\ x_2 + x_3 &= 0. \end{aligned}$$

Choosing $x_4 = t$ and $x_3 = t$ as the free variables, you can describe the solution as $x_1 = 0,\ x_2 = -s,\ x_3 = s,$ and $x_4 = t$, where s and t are any real numbers.

46. The corresponding equations $0 = 0$ and 3 free variables. So, $x_1 = t,\ x_2 = s,\ x_3 = r$, where $t,\ s,\ r$ are any real numbers.

48. (a) If A is the *augmented* matrix of a system of linear equations, then number of equations in this system is three (because it is equal to the number of rows of the augmented matrix). Number of variables is two because it is equal to number of columns of the augmented matrix minus one.

(b) Using Gaussian elimination on the augmented matrix of a system, you have the following.

$$\begin{bmatrix} 2 & -1 & 3 \\ -4 & 2 & k \\ 4 & -2 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -1 & 3 \\ 0 & 0 & k+6 \\ 0 & 0 & 0 \end{bmatrix}$$

This system is consistent if and only if $k + 6 = 0$, so $k = -6$.

(c) If A is the *coefficient* matrix of a system of linear equations, then the number of equations is three, because it is equal the number of rows of the coefficient matrix. The number of variables is also three, because it is equal to the number of columns of the coefficient matrix.

(d) Using Gaussian elimination on A you obtain the following coefficient matrix of an equivalent system.

$$\begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & k+6 \\ 0 & 0 & 0 \end{bmatrix}$$

Because the homogeneous system is always consistent, the homogeneous system with the coefficient matrix A is consistent for any value of k.

50. Using Gaussian elimination on the augmented matrix, you have the following.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ a & b & c & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & (b-a) & c & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & (a-b+c) & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From this row reduced matrix you see that the original system has a unique solution.

52. Because the system composed of equations 1 and 2 is consistent, but has a free variable, this system must have an infinite number of solutions.

54. Use Guass-Jordan elimination as follows.

$$\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 2 & 3\\0 & -3 & -6\\0 & -6 & -12\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 2 & 3\\0 & 1 & 2\\0 & 0 & 0\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 0 & -1\\0 & 1 & 2\\0 & 0 & 0\end{bmatrix}$$

56. Begin by finding all possible first rows

$$[0\ \ 0\ \ 0], [0\ \ 0\ \ 1], [0\ \ 1\ \ 0], [0\ \ 1\ \ a], [1\ \ 0\ \ 0], [1\ \ 0\ \ a], [1\ \ a\ \ b], [1\ \ a\ \ 0],$$

where a and b are nonzero real numbers. For each of these examine the possible remaining rows.

$$\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1 & a\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix},$$

$$\begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0 & 0\\0 & 1 & a\\0 & 0 & 0\end{bmatrix},$$

$$\begin{bmatrix}1 & a & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}, \begin{bmatrix}1 & a & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}, \begin{bmatrix}1 & a & b\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0 & a\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0 & a\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$$

58. (a) False. A 4×7 matrix has 4 rows and 7 columns.

(b) True. Reduced row echelon form of a given matrix is unique while row echelon form is not. (See also exercise 64 of this section.)

(c) True. See Theorem 1.1 on page 25.

(d) False. Multiplying a row by a *nonzero* constant is one of the elementary row operations. However, multiplying a row of a matrix by a constant $c = 0$ is *not* an elementary row operation. (This would change the system by eliminating the equation corresponding to this row.)

60. First you need $a \neq 0$ or $c \neq 0$. If $a \neq 0$, then you have

$$\begin{bmatrix}a & b\\c & d\end{bmatrix} \Rightarrow \begin{bmatrix}a & b\\0 & -\frac{cb}{a} + b\end{bmatrix} \Rightarrow \begin{bmatrix}a & b\\0 & ad - bc\end{bmatrix}.$$

So, $ad - bc = 0$ and $b = 0$, which implies that $d = 0$. If $c \neq 0$, then you interchange rows and proceed.

$$\begin{bmatrix}a & b\\c & d\end{bmatrix} \Rightarrow \begin{bmatrix}c & d\\0 & -\frac{ad}{c} + b\end{bmatrix} \Rightarrow \begin{bmatrix}c & d\\0 & ad - bc\end{bmatrix}$$

Again, $ad - bc = 0$ and $d = 0$, which implies that $b = 0$. In conclusion, $\begin{bmatrix}a & b\\c & d\end{bmatrix}$ is row equivalent to $\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$ if and only if $b = d = 0$, and $a \neq 0$ or $c \neq 0$.

62. Row reduce the augmented matrix for this system.

$$\begin{bmatrix}\lambda - 1 & 2 & 0\\1 & \lambda & 0\end{bmatrix} \Rightarrow \begin{bmatrix}1 & \lambda & 0\\\lambda - 1 & 2 & 0\end{bmatrix} \Rightarrow \begin{bmatrix}1 & \lambda & 0\\0 & (-\lambda^2 + \lambda + 2) & 0\end{bmatrix}$$

To have a nontrivial solution you must have

$$\lambda^2 - \lambda - 2 = 0$$
$$(\lambda - 2)(\lambda + 1) = 0.$$

So, if $\lambda = -1$ or $\lambda = 2$, the system will have nontrivial solutions.

64. No, the echelon form is not unique. For instance, $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The reduced row echelon form is unique.

66. Answers will vary. *Sample answer:* Because the third row consists of all zeros, choose a third equation that is a multiple of one of the other two equations.

$$\begin{aligned} x + 3z &= -2 \\ y + 4z &= 1 \\ 2y + 8z &= 2 \end{aligned}$$

68. When a system of linear equations has infinitely many solutions, the row-echelon form of the corresponding augmented matrix will have a row that is all zeros.

Section 1.3 Applications of Systems of Linear Equations

2. (a) Because there are three points, choose a second-degree polynomial, $p(x) = a_0 + a_1x + a_2x^2$. Then substitute $x = 2, 3,$ and 4 into $p(x)$ and equate the results to $y = 4, 4,$ and 4, respectively.

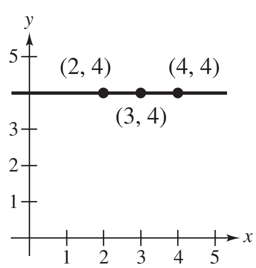
$$\begin{aligned} a_0 + a_1(2) + a_2(2)^2 &= a_0 + 2a_1 + 4a_2 = 4 \\ a_0 + a_1(3) + a_2(3)^2 &= a_0 + 3a_1 + 9a_2 = 4 \\ a_0 + a_1(4) + a_2(4)^2 &= a_0 + 4a_1 + 16a_2 = 4 \end{aligned}$$

Use Gauss-Jordan elimination on the augmented matrix for this system.

$$\begin{bmatrix} 1 & 2 & 4 & 4 \\ 1 & 3 & 9 & 4 \\ 1 & 4 & 16 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

So, $p(x) = 4$.

(b)



4. (a) Because there are four points, choose a third-degree polynomial, $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$. Then substitute $x = -1, 0, 1,$ and 4 into $p(x)$ and equate the results to $y = 3, 0, 1,$ and 58, respectively.

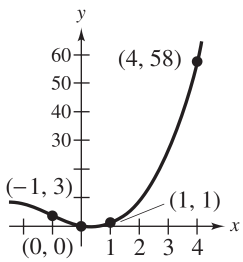
$$\begin{aligned} a_0 + a_1(-1) + a_2(-1)^2 + a_3(-1)^3 &= a_0 - a_1 + a_2 - a_3 = 3 \\ a_0 + a_1(0) + a_2(0)^2 + a_3(0)^3 &= a_0 = 0 \\ a_0 + a_1(1) + a_2(1)^2 + a_3(1)^3 &= a_0 + a_1 + a_2 + a_3 = 1 \\ a_0 + a_1(4) + a_2(4)^2 + a_3(4)^3 &= a_0 + 4a_1 + 16a_2 + 64a_3 = 58 \end{aligned}$$

Use Guass-Jordan elimination on the augmented matrix for this system.

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 3 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 4 & 16 & 64 & 58 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{3}{2} \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}$$

So, $p(x) = -\frac{3}{2}x + 2x^2 + \frac{1}{2}x^3$.

(b)



6. (a) Using the translation $z = x - 2005$, the points (z, y) are (0, 150), (1, 180), (2, 240), and (3, 360). Because there are four points, choose a third-degree polynomial $p(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3$. Then substitute $z = 0, 1, 2,$ and 3 into $p(z)$ and equate the results to $y = 150, 180, 240,$ and 360 respectively.

$$
\begin{aligned}
a_0 + a_1(0) + a_2(0)^2 + a_3(0)^3 &= a_0 &&= 150\\
a_0 + a_1(1) + a_2(1)^2 + a_3(1)^3 &= a_0 + a_1 + a_2 + a_3 &&= 180\\
a_0 + a_1(2) + a_2(2)^2 + a_3(2)^3 &= a_0 + 2a_1 + 4a_2 + 8a_3 &&= 240\\
a_0 + a_1(3) + a_2(3)^2 + a_3(3)^3 &= a_0 + 3a_1 + 9a_2 + 27a_3 &&= 360
\end{aligned}
$$

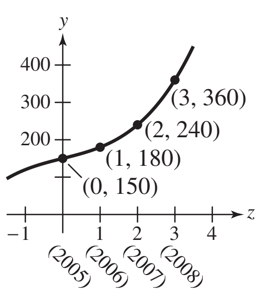
Use Gauss-Jordan elimination on the augmented matrix for this system.

$$
\begin{bmatrix} 1 & 0 & 0 & 0 & 150\\ 1 & 1 & 1 & 1 & 180\\ 1 & 2 & 4 & 8 & 240\\ 1 & 3 & 9 & 27 & 360 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 150\\ 0 & 1 & 0 & 0 & 25\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}
$$

So, $p(z) = 150 + 25z + 5z^3$.

Letting $z = x - 2005$, you have $p(x) = 150 + 25(x - 2005) + 5(x - 2005)^3$.

(b)



8. Letting $p(x) = a_0 + a_1 x + a_2 x^2$, substitute $x = 0, 2,$ and 4 into $p(x)$ and equate the results to $y = 1, \frac{1}{3},$ and $\frac{1}{5}$, respectively.

$$
\begin{aligned}
a_0 + a_1(0) + a_2(0)^2 &= a_0 &&= 1\\
a_0 + a_1(2) + a_2(2)^2 &= a_0 + 2a_1 + 4a_2 &&= \tfrac{1}{3}\\
a_0 + a_1(4) + a_2(4)^2 &= a_0 + 4a_1 + 16a_2 &&= \tfrac{1}{5}
\end{aligned}
$$

Use Gauss-Jordan elimination on the augmented matrix for this system.

$$
\begin{bmatrix} 1 & 0 & 0 & 1\\ 1 & 2 & 4 & \frac{1}{3}\\ 1 & 4 & 16 & \frac{1}{5} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1\\ 0 & 1 & 0 & -\frac{7}{15}\\ 0 & 0 & 1 & \frac{1}{15} \end{bmatrix}
$$

So, $p(x) = 1 - \frac{7}{15}x + \frac{1}{15}x^2$.

10. Let $p(x) = a_0 + a_1x + a_2x^2$ be the equation of the parabola. Because the parabola passes through the points (0, 1) and $\left(\frac{1}{2}, \frac{1}{2}\right)$, you have

$$a_0 + a_1(0) + a_2(0)^2 = a_0 \qquad\qquad = 1$$
$$a_0 + a_1\left(\tfrac{1}{2}\right) + a_2\left(\tfrac{1}{2}\right)^2 = a_0 + \tfrac{1}{2}a_1 + \tfrac{1}{4}a_2 = \tfrac{1}{2}.$$

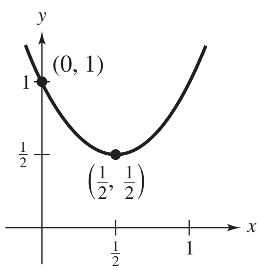
Because $p(x)$ has a horizontal tangent at $\left(\frac{1}{2}, \frac{1}{2}\right)$ the derivative of $p(x)$, $p'(x) = a_1 + 2a_2x$, equals zero when $x = \frac{1}{2}$. So, you have a third linear equation

$$a_1 + 2a_2\left(\tfrac{1}{2}\right) = a_1 + a_2 = 0.$$

Use Gauss-Jordan elimination on the augmented matrix for this linear system.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ 0 & 1 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

So, $p(x) = 1 - 2x + 2x^2$.



12. Assume that the equation of the circle is $x^2 + ax + y^2 + by - c = 0$. Because each of the given points lies on the circle, you have the following linear equations.

$$(1)^2 + a(1) + (3)^2 + b(3) - c = a + 3b - c + 10 = 0$$
$$(-2)^2 + a(-2) + (6)^2 + b(6) - c = -2a + 6b - c + 40 = 0$$
$$(4)^2 + a(4) + (2)^2 + b(2) - c = 4a + 2b - c + 20 = 0$$

Use Gauss-Jordan elimination on the system.

$$\begin{bmatrix} 1 & 3 & -1 & -10 \\ -2 & 6 & -1 & -40 \\ 4 & 2 & -1 & -20 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & -10 \\ 0 & 1 & 0 & -20 \\ 0 & 0 & 1 & -60 \end{bmatrix}$$

So, the equation of the circle is $x^2 - 10x + y^2 - 20y + 60 = 0$ or $(x - 5)^2 + (y - 10)^2 = 65$.

14. (a) Letting

$$z = \frac{(x - 1920)}{10},$$

the four data points are (0, 106), (1, 123), (2, 132), and (3, 151).

Let $p(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3$.

$$\begin{aligned} a_0 \qquad\qquad\qquad &= 106 \\ a_0 + a_1 + a_2 + a_3 &= 123 \\ a_0 + 2a_1 + 4a_2 + 8a_3 &= 132 \\ a_0 + 3a_1 + 9a_2 + 27a_3 &= 151 \end{aligned}$$

The solution to this system is $a_0 = 106$, $a_1 = 27$, $a_2 = -13$, $a_3 = 3$.

So, the cubic polynomial is $p(z) = 106 + 27z - 13z^2 + 3z^3$.

Because

$$z = \frac{(x - 1920)}{10},\ p(x) = 106 + 27\left(\frac{x - 1920}{10}\right) - 13\left(\frac{x - 1920}{10}\right)^2 + 3\left(\frac{x - 1920}{10}\right)^3.$$

(b) To estimate the population in 1960, let $x = 1960$. $p(1960) = 106 + 27(4) - 13(4)^2 + 3(4)^3 = 198$ million. The actual population was 179 million in 1960.

16. (a) Letting $z = x - 2003$, the four points are $(-2, 217.8)$, $(0, 256.3)$, $(2, 312.4)$, and $(4, 377.0)$.

Let $p(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3$.

$$\begin{aligned} a_0 + a_1(-2) + a_2(-2)^2 + a_2(-2)^3 &= a_0 - 2a_1 + 4a_2 - 8a_3 = 217.8 \\ a_0 + a_1(0) + a_2(0)^2 + a_3(0)^3 &= a_0 = 256.3 \\ a_0 + a_1(2) + a_2(2)^2 + a_3(2)^3 &= a_0 + 2a_1 + 4a_2 + 8a_3 = 312.4 \\ a_0 + a_1(4) + a_2(4)^2 + a_3(4)^3 &= a_0 + 4a_1 + 16a_2 + 64a_3 = 377.0 \end{aligned}$$

(b) Use Gauss-Jordan elimination to solve the system.

$$\begin{bmatrix} 1 & -2 & 4 & -8 & 217.8 \\ 1 & 0 & 0 & 0 & 256.3 \\ 1 & 2 & 4 & 8 & 312.4 \\ 1 & 4 & 16 & 64 & 377.0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 256.3 \\ 0 & 1 & 0 & 0 & 24.408\overline{3} \\ 0 & 0 & 1 & 0 & 2.2 \\ 0 & 0 & 0 & 1 & -0.18958\overline{3} \end{bmatrix}$$

So, $p(z) = 256.3 + 24.408\overline{3}z + 2.2z^2 - 0.18958\overline{3}z^3$.

Letting $z = x - 2003$, $p(x) = 256.3 + 24.408\overline{3}(x - 2003) + 2.2(x - 2003)^2 - 0.18958\overline{3}(x - 2003)^3$.

Predicted values after 2007 continue to increase, so the solution produces a reasonable model for predicting future sales.

18. Choosing a second-degree polynomial approximation $p(x) = a_0 + a_1 x + a_2 x^2$, substitute $x = 1, 2,$ and 4 into $p(x)$ and equate the results to $y = 0, 1,$ and 2, respectively.

$$\begin{aligned} a_0 + a_1 + a_2 &= 0 \\ a_0 + 2a_1 + 4a_2 &= 1 \\ a_0 + 4a_1 + 16a_2 &= 2 \end{aligned}$$

The solution to this system is $a_0 = -\frac{4}{3}$, $a_1 = \frac{3}{2}$, and $a_2 = -\frac{1}{6}$.

So, $p(x) = -\frac{4}{3} + \frac{3}{2}x - \frac{1}{6}x^2$.

Finally, to estimate $\log_2 3$, calculate $p(3) = -\frac{4}{3} + \frac{3}{2}(3) - \frac{1}{6}(3)^2 = \frac{5}{3}$.

20. Let

$p_1(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1}$ and

$p_2(x) = b_0 + b_1x + b_2x^2 + \cdots + b_{n-1}x^{n-1}$

be two different polynomials that pass through the n given points. The polynomial

$p_1(x) - p_2(x) = (a_0 - b_0) + (a_1 - b_1)x + (a_2 - b_2)x^2 + \cdots + (a_{n-1} - b_{n-1})x^{n-1}$

is zero for these n values of x. So $a_0 = b_0, a_1 = b_1, a_2 = b_2, \ldots, a_{n-1} = b_{n-1}$.

Therefore, there is only one polynomial function of degree $n - 1$ (or less) whose graph passes through n points in the plane with distinct x-coordinates.

22. (a) Each of the network's four junctions gives rise to a linear equation as shown below.

$$\begin{aligned} \textbf{input} &= \textbf{output} \\ 300 &= x_1 + x_2 \\ x_1 + x_3 &= x_4 + 150 \\ x_2 + 200 &= x_3 + x_5 \\ x_4 + x_5 &= 350 \end{aligned}$$

Reorganize these equations, form the augmented matrix, and use Gauss-Jordan elimination.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 300 \\ 1 & 0 & 1 & -1 & 0 & 150 \\ 0 & 1 & -1 & 0 & -1 & -200 \\ 0 & 0 & 0 & 1 & 1 & 350 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 500 \\ 0 & 1 & -1 & 0 & -1 & -200 \\ 0 & 0 & 0 & 1 & 1 & 350 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Letting $x_5 = t$ and $x_3 = s$ be the free variables, you have

$x_1 = 500 - s - t$

$x_2 = -200 + s + t$

$x_3 = s$

$x_4 = 350 - t$

$x_5 = t.$

(b) If $x_2 = 200$ and $x_3 = 50$, then you have $s = 50$ and $t = 350$.

So, the solution is: $x_1 = 100,\ x_2 = 200,\ x_3 = 50,\ x_4 = 0,\ x_5 = 350.$

(c) If $x_2 = 150$ and $x_3 = 0$, then you have $s = 0$ and $t = 350$.

So, the solution is: $x_1 = 150,\ x_2 = 150,\ x_3 = 0,\ x_4 = 0,\ x_5 = 350.$

24. (a) Each of the network's four junctions gives rise to a linear equation as shown below.

$$\begin{aligned} \textbf{input} &= \textbf{output} \\ 400 + x_2 &= x_1 \\ x_1 + x_3 &= x_4 + 600 \\ 300 &= x_2 + x_3 + x_5 \\ x_4 + x_5 &= 100 \end{aligned}$$

Reorganize these equations, form the augmented matrix, and use Gauss-Jordan elimination.

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 400 \\ 1 & 0 & 1 & -1 & 0 & 600 \\ 0 & 1 & 1 & 0 & 1 & 300 \\ 0 & 0 & 0 & 1 & 1 & 100 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 700 \\ 0 & 1 & 1 & 0 & 1 & 300 \\ 0 & 0 & 0 & 1 & 1 & 100 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Letting $x_5 = t$ and $x_3 = s$ be the free variables, you can describe the solution as

$x_1 = 700 - s - t$

$x_2 = 300 - s - t$

$x_3 = s$

$x_4 = 100 - t$

$x_5 = t$, where t and s are any real numbers.

(b) If $x_3 = 0$ and $x_5 = 100$, then the solution is: $x_1 = 600,\ x_2 = 200,\ x_3 = 0,\ x_4 = 0,\ x_5 = 100.$

(c) If $x_3 = x_5 = 100$, then the solution is: $x_1 = 500,\ x_2 = 100,\ x_3 = 100,\ x_4 = 0,\ x_5 = 100.$

26. Applying Kirchoff's first law to either junction produces

$I_1 + I_3 = I_2$

and applying the second law to the two paths produces

$$R_1 I_1 + R_2 I_2 = 4I_1 + I_2 = 16$$
$$R_2 I_2 + R_3 I_3 = I_2 + 4I_3 = 8.$$

Rearrange these equations, form the augmented matrix, and use Gauss-Jordan elimination.

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 4 & 1 & 0 & 16 \\ 0 & 1 & 4 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

So, the solution is: $I_1 = 3$, $I_2 = 4$, and $I_3 = 1$.

28. Applying Kirchoff's first law to three of the four junctions produces

$$I_1 + I_3 = I_2$$
$$I_1 + I_4 = I_2$$
$$I_3 + I_6 = I_5$$

and applying the second law to the three paths produces

$$R_1 I_1 + R_2 I_2 = 3I_1 + 2I_2 = 14$$
$$R_2 I_2 + R_4 I_4 + R_5 I_5 + R_3 I_3 = 2I_2 + 2I_4 + I_5 + 4I_3 = 25$$
$$R_5 I_5 + R_6 I_6 = I_5 + I_6 = 8.$$

Rearrange these equations, form the augmented matrix, and use Gauss-Jordan elimination.

$$\begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 3 & 2 & 0 & 0 & 0 & 0 & 14 \\ 0 & 2 & 4 & 2 & 1 & 0 & 25 \\ 0 & 0 & 0 & 0 & 1 & 1 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

So, the solution is: $I_1 = 2$, $I_2 = 4$, $I_3 = 2$, $I_4 = 2$, $I_5 = 5$, and $I_6 = 3$.

30. $\dfrac{8x^2}{(x-1)^2(x+1)} = \dfrac{A}{x+1} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2}$

$$8x^2 = A(x-1)^2 + B(x-1)(x+1) + C(x+1)$$
$$8x^2 = Ax^2 - 2Ax + A + Bx^2 - B + Cx + C$$
$$8x^2 = (A+B)x^2 + (-2A+C)x + A - B + C$$

So,
$$\begin{aligned} A + B \phantom{{}+C} &= 8 \\ -2A \phantom{{}+B} + C &= 0 \\ A - B + C &= 0. \end{aligned}$$

Use Gauss-Jordan elimination to solve the system.

$$\begin{bmatrix} 1 & 1 & 0 & 8 \\ -2 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

The solution is: $A = 2$, $B = 6$, and $C = 4$.

So, $\dfrac{8x^2}{(x-1)^2(x+1)} = \dfrac{2}{x+1} + \dfrac{6}{x-1} + \dfrac{4}{(x-1)^2}.$

32. $\dfrac{3x^2 - 7x - 12}{(x+4)(x-4)^2} = \dfrac{A}{x+4} + \dfrac{B}{x-4} + \dfrac{C}{(x-4)^2}$

$$3x^2 - 7x - 12 = A(x-4)^2 + B(x+4)(x-4) + C(x+4)$$
$$3x^2 - 7x - 12 = Ax^2 - 8Ax + 16A + Bx^2 - 16B + Cx + 4C$$
$$3x^2 - 7x - 12 = (A+B)x^2 + (-8A+C)x + 16A - 16B + 4C$$

So,
$$\begin{aligned} A + B \quad &= 3 \\ -8A \quad + C &= -7 \\ 16A - 16B + 4C &= -12. \end{aligned}$$

Use Gauss-Jordan elimination to solve the system.

$$\begin{bmatrix} 1 & 1 & 0 & 3 \\ -8 & 0 & 1 & -7 \\ 16 & -16 & 4 & -12 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The solution is: $A = 1$, $B = 2$, and $C = 1$.

So, $\dfrac{3x^2 - 7x - 12}{(x+4)(x-4)^2} = \dfrac{1}{x+4} + \dfrac{2}{x-4} + \dfrac{1}{(x-4)^2}$

34. Use Gauss-Jordan elimination to solve the system.

$$\begin{bmatrix} 0 & 2 & 2 & -2 \\ 2 & 0 & 1 & -1 \\ 2 & 1 & 0 & 100 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 25 \\ 0 & 1 & 0 & 50 \\ 0 & 0 & 1 & -51 \end{bmatrix}$$

So, $x = 25$, $y = 50$, and $\lambda = -51$.

36. Let $x =$ number of touchdowns, $y =$ number of extra-point kicks, and $z =$ number of field goals.

$$\begin{aligned} 6x + y + 3z &= 55 \\ x - y \quad &= 0 \\ x \quad - 3z &= 1 \end{aligned}$$

Use Gauss-Jordan elimination to solve the system.

$$\begin{bmatrix} 6 & 1 & 3 & 55 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -3 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Because $x = 7$, $y = 7$, and $x = 2$, there were 7 touchdowns, 7 extra-point kicks, and 2 field goals.

Review Exercises for Chapter 1

2. Because the equation cannot be written in the form $a_1x + a_2y = b$, it is *not* linear in the variables *x* and *y*.

4. Because the equation is in the form $a_1x + a_2y = b$, it *is* linear in the variables *x* and *y*.

6. Because the equation cannot be written in the form $a_1x + a_2y = b$, it is *not* linear in the variables *x* and *y*.

8. Because the equation is in the form $a_1x + a_2x = b$, it *is* linear in the variables *x* and *y*.

10. Choosing x_2 and x_3 as the free variables and letting $x_2 = s$ and $x_3 = t$, you have

$$\begin{aligned} 3x_1 + 2s - 4t &= 0 \\ 3x_1 &= -2s + 4t \\ x_1 &= \tfrac{1}{3}(-2s + 4t). \end{aligned}$$

So, the solution set can be described as $x_1 = -\frac{2}{3}s + \frac{4}{3}t$, $x_2 = s$, and $x_3 = t$, where *s* and *t* are real numbers.

12. Row reduce the augmented matrix for this system.

$$\begin{bmatrix} 1 & 1 & -1 \\ 3 & 2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \end{bmatrix}$$

Converting back to a linear system, the solution is $x = 2$ and $y = -3$.

14. Rearrange the equations, form the augmented matrix, and row reduce.

$$\begin{bmatrix}1 & -1 & 3\\ 4 & -1 & 10\end{bmatrix} \Rightarrow \begin{bmatrix}1 & -1 & 3\\ 0 & 3 & -2\end{bmatrix} \Rightarrow \begin{bmatrix}1 & -1 & 3\\ 0 & 1 & -\frac{2}{3}\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 0 & \frac{7}{3}\\ 0 & 1 & -\frac{2}{3}\end{bmatrix}.$$

Converting back to a linear system, you obtain the solution $x = \frac{7}{3}$ and $y = -\frac{2}{3}$.

16. Rearrange the equations, form the augmented matrix, and row reduce.

$$\begin{bmatrix}4 & 1 & 0\\ -1 & 1 & 0\end{bmatrix} \Rightarrow \begin{bmatrix}1 & -1 & 0\\ 4 & 1 & 0\end{bmatrix} \Rightarrow \begin{bmatrix}1 & -1 & 0\\ 0 & 5 & 0\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}.$$

Converting back to a linear system, the solution is $x = y = 0$.

18. Row reduce the augmented matrix for this system.

$$\begin{bmatrix}40 & 30 & 24\\ 20 & 15 & -14\end{bmatrix} \Rightarrow \begin{bmatrix}1 & \frac{3}{4} & \frac{3}{5}\\ 20 & 15 & -14\end{bmatrix} \Rightarrow \begin{bmatrix}1 & \frac{3}{4} & \frac{3}{5}\\ 0 & 0 & -26\end{bmatrix}$$

Because the second row corresponds to the false statement $0 = -26$, the system has no solution.

20. Multiplying both equations by 100 and forming the augmented matrix produces

$$\begin{bmatrix}20 & -10 & 7\\ 40 & -50 & -1\end{bmatrix}.$$

Gauss-Jordan elimination yields the following.

$$\begin{bmatrix}1 & -\frac{1}{2} & \frac{7}{20}\\ 40 & -50 & -1\end{bmatrix} \Rightarrow \begin{bmatrix}1 & -\frac{1}{2} & \frac{7}{20}\\ 0 & -30 & -15\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}1 & -\frac{1}{2} & \frac{7}{20}\\ 0 & 1 & \frac{1}{2}\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 0 & \frac{3}{5}\\ 0 & 1 & \frac{1}{2}\end{bmatrix}$$

So, the solution is: $x = \frac{3}{5}$ and $y = \frac{1}{2}$.

22. Use Gauss-Jordan elimination on the augmented matrix.

$$\begin{bmatrix}\frac{1}{3} & \frac{4}{7} & 3\\ 2 & 3 & 15\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 0 & -3\\ 0 & 1 & 7\end{bmatrix}$$

So, the solution is: $x = -3$, $y = 7$.

24. Because the matrix has 3 rows and 2 columns, it has size 3×2.

26. The matrix satisfies all three conditions in the definition of row-echelon form. Because each column that has a leading 1 (columns 1 and 4) has zeros elsewhere, the matrix is in reduced row-echelon form.

28. The matrix satisfies all three conditions in the definition of row-echelon form. Because each column that has a leading 1 (columns 2 and 3) has zeros elsewhere, the matrix is in reduced row-echelon form.

30. This matrix corresponds to the system

$$x_1 + 2x_2 + 3x_3 = 0$$
$$0 = 1.$$

Because the second equation is impossible, the system has no solution.

32. Use Gauss-Jordan elimination on the augmented matrix.

$$\begin{bmatrix}2 & 3 & 1 & 10\\ 2 & -3 & -3 & 22\\ 4 & -2 & 3 & -2\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 0 & 0 & 5\\ 0 & 1 & 0 & 2\\ 0 & 0 & 1 & -6\end{bmatrix}$$

So, the solution is: $x = 5$, $y = 2$, and $z = -6$.

34. Use Gauss-Jordan elimination on the augmented matrix.

$$\begin{bmatrix}2 & 0 & 6 & -9\\ 3 & -2 & 11 & -16\\ 3 & -1 & 7 & -11\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 0 & 0 & -\frac{3}{4}\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & -\frac{5}{4}\end{bmatrix}$$

So, the solution is: $x = -\frac{3}{4}$, $y = 0$, and $z = -\frac{5}{4}$.

36. Use Gauss-Jordan elimination on the augmented matrix.

$$\begin{bmatrix}1 & 2 & 6 & 1\\ 2 & 5 & 15 & 4\\ 3 & 1 & 3 & -6\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 2 & 6 & 1\\ 0 & 1 & 3 & 2\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Because the third row corresponds to the false statement $0 = 1$, there is no solution.

38. Use Gauss-Jordan elimination on the augmented matrix.

$$\begin{bmatrix}2 & 5 & -19 & 34\\ 3 & 8 & -31 & 54\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 0 & 3 & 2\\ 0 & 1 & -5 & 6\end{bmatrix}$$

So, the solution is: $x_1 = 2 - 3t$, $x_2 = 6 + 5t$, and $x_3 = t$, where t is any real number.

40. Use Gauss-Jordan elimination on the augmented matrix.

$$\begin{bmatrix}1 & 5 & 3 & 0 & 0 & 14\\ 0 & 4 & 2 & 5 & 0 & 3\\ 0 & 0 & 3 & 8 & 6 & 16\\ 2 & 4 & 0 & 0 & -2 & 0\\ 2 & 0 & -1 & 0 & 0 & 0\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 0 & 0 & 0 & 0 & 2\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 4\\ 0 & 0 & 0 & 1 & 0 & -1\\ 0 & 0 & 0 & 0 & 1 & 2\end{bmatrix}$$

So, the solution is: $x_1 = 2$, $x_2 = 0$, $x_3 = 4$, $x_4 = -1$, $x_5 = 2$.

42. Using a graphing utility, the augmented matrix reduces to

$$\begin{bmatrix} 1 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The system is inconsistent so there is no solution.

44. Using a graphing utility, the augmented matrix reduces to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{407}{98} \\ 0 & 1 & 0 & 0 & -\frac{335}{49} \\ 0 & 0 & 1 & 0 & -\frac{3}{2} \\ 0 & 0 & 0 & 1 & \frac{743}{98} \end{bmatrix}.$$

So, the solution is: $x = -\frac{407}{98}$, $y = -\frac{335}{49}$, $z = -\frac{3}{2}$, and $w = \frac{743}{98}$.

46. Using a graphing utility, the augmented matrix reduces to

$$\begin{bmatrix} 1 & 0 & 0 & 1.5 & 0 \\ 0 & 1 & 0 & 0.5 & 0 \\ 0 & 0 & 1 & 0.5 & 0 \end{bmatrix}.$$

Letting $w = t$, you have $x = -1.5t$, $y = -0.5t$, $z = -0.5t$, $w = t$, where t is any real number.

48. Use Gauss-Jordan elimination on the augmented matrix.

$$\begin{bmatrix} 2 & 4 & -7 & 0 \\ 1 & -3 & 9 & 0 \\ 6 & 0 & 9 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & \frac{3}{2} & 0 \\ 0 & 1 & -\frac{5}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Letting $x_3 = t$ be the free variable, you have $x_1 = -\frac{3}{2}t$, $x_2 = \frac{5}{2}t$, and $x_3 = t$, where t is any real number.

50. Use Gauss-Jordan elimination on the augmented matrix.

$$\begin{bmatrix} 1 & 3 & 5 & 0 \\ 1 & 4 & \frac{1}{2} & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & \frac{37}{2} & 0 \\ 0 & 1 & -\frac{9}{2} & 0 \end{bmatrix}$$

Letting $x_3 = t$ be the free variable, you have $x_1 = -\frac{37}{2}t$, $x_2 = \frac{9}{2}t$, and $x_3 = t$.

52. Use Gaussian elimination on the augmented matrix.

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ -1 & 1 & -1 & 0 \\ 1 & k & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & (k+1) & -1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & (k+1) & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

So, there will be exactly one solution (the trivial solution $x = y = z = 0$) if and only if $k \neq -1$.

54. Form the augmented matrix for the system

$$\begin{bmatrix} 2 & -1 & 1 & a \\ 1 & 1 & 2 & b \\ 0 & 3 & 3 & c \end{bmatrix}$$

and use Gaussian elimination to reduce the matrix to row-echelon form.

$$\begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & \frac{a}{2} \\ 1 & 1 & 2 & b \\ 0 & 3 & 3 & c \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{a}{2} \\ 0 & \frac{3}{2} & \frac{3}{2} & \frac{2b-a}{2} \\ 0 & 3 & 3 & c \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & \frac{a}{2} \\ 0 & 1 & 1 & \frac{2b-a}{3} \\ 0 & 3 & 3 & c \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & \frac{a}{2} \\ 0 & 1 & 1 & \frac{2b-a}{3} \\ 0 & 0 & 0 & c-2b+a \end{bmatrix}$$

(a) If $c - 2b + a \neq 0$, then the system has no solution.

(b) The system cannot have one solution.

(c) If $c - 2b + a = 0$, then the system has infinitely many solutions.

56. Find all possible first rows.

$[0\ \ 0\ \ 0], [0\ \ 0\ \ 1], [0\ \ 1\ \ 0], [0\ \ 1\ \ a], [1\ \ 0\ \ 0], [1\ \ a\ \ 0], [1\ \ a\ \ b], [1\ \ 0\ \ a]$

where a and b are nonzero real numbers. For each of these examine the possible second rows.

$$\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&1\end{bmatrix},$$

$$\begin{bmatrix}0&1&a\\0&0&0\end{bmatrix}, \begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}, \begin{bmatrix}1&0&0\\0&0&1\end{bmatrix}, \begin{bmatrix}1&0&0\\0&1&a\end{bmatrix},$$

$$\begin{bmatrix}1&a&0\\0&0&0\end{bmatrix}, \begin{bmatrix}1&a&0\\0&0&1\end{bmatrix}, \begin{bmatrix}1&a&b\\0&0&0\end{bmatrix}, \begin{bmatrix}1&0&a\\0&0&0\end{bmatrix}, \begin{bmatrix}1&0&a\\0&1&0\end{bmatrix}$$

58. Use Gaussian elimination on the augmented matrix.

$$\begin{bmatrix}(\lambda+2)&-2&3&0\\-2&(\lambda-1)&6&0\\1&2&\lambda&0\end{bmatrix} \Rightarrow \begin{bmatrix}1&2&\lambda&0\\0&\lambda+3&6+2\lambda&0\\0&-2\lambda-6&-\lambda^2-2\lambda+3&0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}1&2&\lambda&0\\0&\lambda+3&6+2\lambda&0\\0&0&(\lambda^2-2\lambda-15)&0\end{bmatrix}$$

So, you need $\lambda^2 - 2\lambda - 15 = (\lambda - 5)(\lambda + 3) = 0$, which implies $\lambda = 5$ or $\lambda = -3$.

60. (a) True. A homogeneous system of linear equations is always consistent, because there is always a trivial solution, *i.e.*, when all variables are equal to zero. See Theorem 1.1 on page 25.

(b) False. Consider for example the following system (with three variables and two equations).

$$\begin{aligned} x + y - z &= 2\\ -2x - 2y + 2z &= 1. \end{aligned}$$

It is easy to see that this system has *no* solutions.

62. (a) Let x = number of touchdowns, y = number of extra-point kicks, and z = number of field goals.

$$\begin{aligned} 6x + y + 3z &= 45\\ x - y &= 0\\ x - 6z &= 0 \end{aligned}$$

(b) Use Gauss-Jordan elimination to solve the system.

$$\begin{bmatrix}6&1&3&45\\1&-1&0&0\\1&0&-6&0\end{bmatrix} \Rightarrow \begin{bmatrix}1&0&0&6\\0&1&0&6\\0&0&1&1\end{bmatrix}$$

Because $x = 6$, $y = 6$, and $z = 1$, there were 6 touchdowns, 6 extra-point kicks, and 1 field goal.

64. $\dfrac{3x^2 + 3x - 2}{(x+1)^2(x-1)} = \dfrac{A}{x+1} + \dfrac{B}{x-1} + \dfrac{C}{(x+1)^2}$

$$3x^2 + 3x - 2 = A(x+1)(x-1) + B(x+1)^2 + C(x-1)$$
$$3x^2 + 3x - 2 = Ax^2 - A + Bx^2 + 2Bx + B + Cx - C$$
$$3x^2 + 3x - 2 = (A+B)x^2 + (2B+C)x - A + B - C$$

So,
$$\begin{aligned} A + B \quad &= 3 \\ 2B + C &= 3 \\ -A + B - C &= -2. \end{aligned}$$

Use Gauss-Jordan elimination to solve the system.

$$\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 2 & 1 & 3 \\ -1 & 1 & -1 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The solution is: $A = 2$, $B = 1$, and $C = 1$.

So, $\dfrac{3x^2 + 3x - 2}{(x+1)^2(x-1)} = \dfrac{2}{x+1} + \dfrac{1}{x-1} + \dfrac{1}{(x+1)^2}$.

66. (a) Because there are four points, choose a third-degree polynomial, $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$.

By substituting the values at each point into this equation, you obtain the system

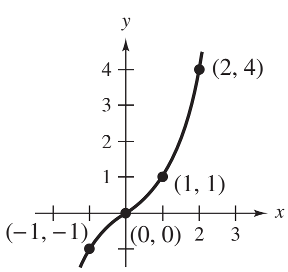
$$\begin{aligned} a_0 - a_1 + a_2 - a_3 &= -1 \\ a_0 \quad &= 0 \\ a_0 + a_1 + a_2 + a_3 &= 1 \\ a_0 + 2a_1 + 4a_2 + 8a_3 &= 4. \end{aligned}$$

Use Gauss-Jordan elimination on the augmented matrix.

$$\begin{bmatrix} 1 & -1 & 1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{2}{3} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{3} \end{bmatrix}$$

So, $p(x) = \frac{2}{3}x + \frac{1}{3}x^3$.

(b)



68. Substituting the points, (1, 0), (2, 0), (3, 0), and (4, 0) into the polynomial $p(x)$ yields the system

$$\begin{aligned} a_0 + a_1 + a_2 + a_3 &= 0 \\ a_0 + 2a_1 + 4a_2 + 8a_3 &= 0 \\ a_0 + 3a_1 + 9a_2 + 27a_3 &= 0 \\ a_0 + 4a_1 + 16a_2 + 64a_3 &= 0. \end{aligned}$$

Gaussian elimination shows that the only solution is $a_0 = a_1 = a_2 = a_3 = 0$.

70. You are looking for a quadratic function $y = ax^2 + bx + c$, such that the points (0, 40), (6, 73) and (12, 52) lie on its graph. The system of equations to fit the data to a quadratic polynomial is

$$\begin{aligned} c &= 40 \\ 36a + 6b + c &= 73 \\ 144a + 12b + c &= 52. \end{aligned}$$

Substituting $c = 40$ into the second and the third equations, you obtain

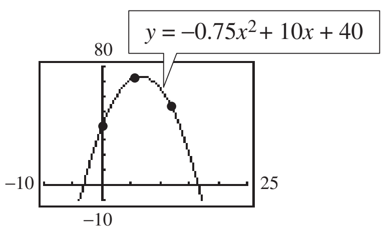
$$\begin{aligned} 36a + 6b &= 33 \\ 144a + 12b &= 12 \end{aligned}$$

or

$$\begin{aligned} 12a + 2b &= 11 \\ 12a + b &= 1. \end{aligned}$$

Adding -1 times the second equation to the first equation you obtain $b = 10$, so $a = -\frac{9}{12} = -\frac{3}{4}$.

If you use the graphing utility to graph $y = -0.75x^2 + 10x + 40$ you would see that the data points (0, 40), (6, 73) and (12, 52) lie on the graph. On Page 29 it is stated that if we have n points with district x-coordinates, then there is precisely one polynomial function of degree $n - 1$ (or less) that fits these points.



72. (a) First find the equations corresponding to each node in the network.

input = output

$$\begin{aligned} x_1 + 200 &= x_2 + x_4 \\ x_6 + 100 &= x_1 + x_3 \\ x_2 + x_3 &= x_5 + 300 \\ x_4 + x_5 &= x_6 \end{aligned}$$

Rearranging this system and forming the augmented matrix, you have

$$\begin{bmatrix} 1 & -1 & 0 & -1 & 0 & 0 & -200 \\ 1 & 0 & 1 & 0 & 0 & -1 & 100 \\ 0 & 1 & 1 & 0 & -1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 \end{bmatrix}.$$

The equivalent reduced row-echelon matrix is

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & -1 & 100 \\ 0 & 1 & 1 & 0 & -1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Choosing $x_3 = r$, $x_5 = s$, and $x_6 = t$ as the free variables, you obtain

$$\begin{aligned} x_1 &= 100 - r + t \\ x_2 &= 300 - r + s \\ x_4 &= -s + t, \end{aligned}$$

where r, s, and t are any real numbers.

(b) When $x_3 = 100 = r$, $x_5 = 50 = s$, and $x_6 = 50 = t$, you have

$$\begin{aligned} x_1 &= 100 - 100 + 50 = 50 \\ x_2 &= 300 - 100 + 50 = 250 \\ x_4 &= -50 + 50 = 0. \end{aligned}$$

Project Solutions for Chapter 1

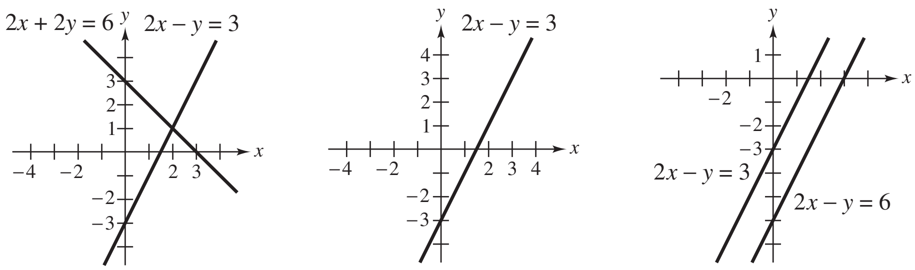
1 Graphing Linear Equations

1. $\begin{bmatrix} 2 & -1 & 3 \\ a & b & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & b + \frac{1}{2}a & 6 - \frac{3}{2}a \end{bmatrix}$

(a) Unique solution if $b + \frac{1}{2}a \neq 0$. For instance, $a = b = 2$.

(b) Infinite number of solutions if $b + \frac{1}{2}a = 6 - \frac{3}{2}a = 0 \Rightarrow a = 4$ and $b = -2$.

(c) No solution if $b + \frac{1}{2}a = 0$ and $6 - \frac{3}{2}a \neq 0 \Rightarrow a \neq 4$ and $b = -\frac{1}{2}a$. For instance, $a = 2$, $b = -1$.

(d)


(a) $\begin{aligned} 2x - y &= 3 \\ 2x + 2y &= 6 \end{aligned}$ (b) $\begin{aligned} 2x - y &= 3 \\ 4x - 2y &= 6 \end{aligned}$ (c) $\begin{aligned} 2x - y &= 3 \\ 2x - y &= 6 \end{aligned}$

(The answers are not unique.)

2. (a) $\begin{aligned} x + y + z &= 0 \\ x + y + z &= 0 \\ x - y - z &= 0 \end{aligned}$ (b) $\begin{aligned} x + y + z &= 0 \\ y + z &= 1 \\ z &= 2 \end{aligned}$ (c) $\begin{aligned} x + y + z &= 0 \\ x + y + z &= 1 \\ x - y - z &= 0 \end{aligned}$

(The answers are not unique.)

There are other configurations, such as three mutually parallel planes, or three planes that intersect pairwise in lines.

2 Underdetermined and Overdetermined Systems of Equations

1. Yes, $x + y = 2$ is a consistent underdetermined system.

2. Yes,

$$\begin{aligned} x + y &= 2 \\ 2x + 2y &= 4 \\ 3x + 3y &= 6 \end{aligned}$$

is a consistent, overdetermined system.

3. Yes,

$$\begin{aligned} x + y + z &= 1 \\ x + y + z &= 2 \end{aligned}$$

is an inconsistent underdetermined system.

4. Yes,

$$\begin{aligned} x + y &= 1 \\ x + y &= 2 \\ x + y &= 3 \end{aligned}$$

is an inconsistent underdetermined system.

5. In general, a linear system with more equations than variables would probably be inconsistent. Here is an intuitive reason: Each variable represents a degree of freedom, while each equation gives a condition that in general reduces number of degrees of freedom by one. If there are more equations (conditions) than variables (degrees of freedom), then there are too many conditions for the system to be consistent. So you expect such a system to be inconsistent in general. But, as Exercise 2 shows, this is not always true.

6. In general, a linear system with more variables than equations would probably be consistent. As in Exercise 5 the intuitive explanation is as follows. Each variable represents a degree of freedom, and each equation represents a condition that takes away one degree of freedom. If there are more variables than equations, in general, you would expect a solution. But, as Exercise 3 shows, this is not always true.