Challenge Problems

Chapter 3

A Click here for answers. S Click here for solutions.

1. (a) Find the domain of the function $f(x) = \sqrt{1 - \sqrt{2 - \sqrt{3 - x}}}$.
(b) Find $f'(x)$.
(c) Check your work in parts (a) and (b) by graphing f and f' on the same screen.

Chapter 4

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1. Find the absolute maximum value of the function

$$f(x) = \frac{1}{1 + |x|} + \frac{1}{1 + |x - 2|}$$

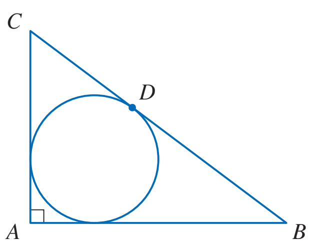

FIGURE FOR PROBLEM 2

2. (a) Let ABC be a triangle with right angle A and hypotenuse $a = |BC|$. (See the figure.) If the inscribed circle touches the hypotenuse at D, show that

$$|CD| = \tfrac{1}{2}(|BC| + |AC| - |AB|)$$

(b) If $\theta = \frac{1}{2}\angle C$, express the radius r of the inscribed circle in terms of a and θ.
(c) If a is fixed and θ varies, find the maximum value of r.

3. A triangle with sides a, b, and c varies with time t, but its area never changes. Let θ be the angle opposite the side of length a and suppose θ always remains acute.
(a) Express $d\theta/dt$ in terms of b, c, θ, db/dt, and dc/dt.
(b) Express da/dt in terms of the quantities in part (a).

Chapter 5

A Click here for answers. S Click here for solutions.

1. In Sections 5.1 and 5.2 we used the formulas for the sums of the kth powers of the first n integers when $k = 1, 2$, and 3. (These formulas are proved in Appendix E.) In this problem we derive formulas for any k. These formulas were first published in 1713 by the Swiss mathematician James Bernoulli in his book *Ars Conjectandi*.
(a) The **Bernoulli polynomials** B_n are defined by $B_0(x) = 1$, $B_n'(x) = B_{n-1}(x)$, and $\int_0^1 B_n(x)\,dx = 0$ for $n = 1, 2, 3, \ldots$. Find $B_n(x)$ for $n = 1, 2, 3$, and 4.
(b) Use the Fundamental Theorem of Calculus to show that $B_n(0) = B_n(1)$ for $n \geqslant 2$.
(c) If we introduce the **Bernoulli numbers** $b_n = n!\,B_n(0)$, then we can write

$$B_0(x) = b_0 \qquad B_1(x) = \frac{x}{1!} + \frac{b_1}{1!}$$

$$B_2(x) = \frac{x^2}{2!} + \frac{b_1}{1!}\frac{x}{1!} + \frac{b_2}{2!} \qquad B_3(x) = \frac{x^3}{3!} + \frac{b_1}{1!}\frac{x^2}{2!} + \frac{b_2}{2!}\frac{x}{1!} + \frac{b_3}{3!}$$

and, in general,

$$B_n(x) = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k} b_k x^{n-k} \qquad \text{where} \qquad \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

[The numbers $\binom{n}{k}$ are the binomial coefficients.] Use part (b) to show that, for $n \geqslant 2$,

$$b_n = \sum_{k=0}^{n}\binom{n}{k} b_k$$