Chapter 1

Fundamentals

1 Joy

This section may be assigned as reading. The purpose of this section is to instill in students a sense of the pleasure mathematical work can bring. I recommend that you assign the one problem contained herein but not for course credit. Admonish your students thoroughly not to discuss the problem with each other or else they will spoil the experience.

1.1 Do not, under any circumstances, tell students the answer to this problem or you will rob them of the joy of discovering the answer themselves. Do not give them hints. Do not ask leading questions such as "What is the 24^{th} factor in the expression?"

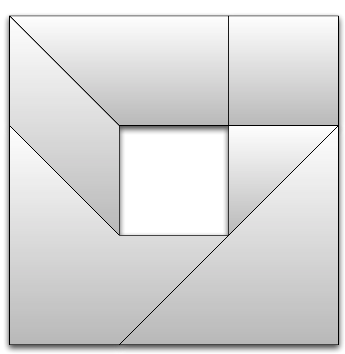
Let students work this out for themselves. Be encouraging and reassure them that when they have found the answer, they will know they are correct.

2 Speaking (and Writing) of Mathematics

This section may also be assigned as reading. The goal here is to emphasize that clarity of language is vital in mathematics. The extent to which students can articulate their thoughts is a good indicator of how well they understand them. Furthermore, the very act of putting their thoughts into clear sentences helps with the learning process.

Students often have the unhealthy notion that to do mathematics is to "grunt out" a series of equations with no words of explanation. They also believe that mathematics instructors will accept lousy writing, horrible penmanship, and innumerable crossouts on crumpled paper that has been torn out of a spiral notebook. We need to expect a lot better!

2.1 This diagram gives a solution to the puzzle.



Here are written instructions.

You will find before you six pieces: a large isosceles right triangle, a small isosceles right triangle, a square, a parallelogram, a trapezoid (with one side perpendicular to the parallel sides), and an oddly shaped (nonconvex) pentagon.

1. Place the square in the upper right with its sides vertical and horizontal.
2. Place the trapezoid to the left of the square so that its long parallel side aligns with the top of the square and its short parallel side aligns with the bottom of the square. Note that the side perpendicular to the parallel sides exactly abuts the left side of the square.
3. Place the parallelogram so that one of the long sides of the parallelogram matches the long non-parallel side of the trapezoid. Thus, the parallelogram is just below the left portion of the trapezoid. On the left, the short side of the parallelogram should complete a right angle with the long parallel side of the trapezoid.
4. Place the small right triangle below the square. One of the legs of the small right triangle exactly aligns with the bottom of the square and the other leg of the right triangle faces the inside of the puzzle. The hypotenuse of the small right triangle should slope from the lower left to the upper right.
5. Hold the large right triangle so that its legs are vertical and horizontal, and its right angle is to the lower right. Now place the large right triangle so that its upper 45° angle just touches the lower right corner of the square. Half of the hypotenuse of the large right triangle should align with all of the hypotenuse of the small right triangle.
6. Finally, hold the pentagon so that its right angle is in the lower left and the two sides that form that right angle are vertical and horizontal. It will now slide so that its right edge matches the lower half of the hypotenuse of the big right triangle and its upper edge meets up with the lower, long side of the parallelogram.

3 Definition

If we think of mathematics as a dramatic production, there are three main characters: Definition, Theorem, and Proof. (We can think of Counterexample as being the "evil" side of Proof. Important, supporting roles in this show are Conjecture and Example.) Thus the first few sections of this book are dedicated to introducing these main characters.

Perhaps conspicuous by its absence is Axiom. (The term is introduced later in the book.) This omission is intentional. In my philosophy of mathematics, axioms are a form of definition. For example, the axioms of Euclidean geometry form the *definition* of the Euclidean plane. Any collection of objects we call *points* and *lines* that satisfy the Euclidean axioms is "honored" with the title *Euclidean plane*. Likewise the so-called axioms of group theory are actually the definition of a group. I have found that introducing axioms as "unproved assumptions" undermines the truth and beauty of mathematics. It unnecessarily introduces some doubt and confusion in students' minds.

And while we are speaking of philosophy, I try to be rather clear about mine in the text. To me, mathematics is a purely mental construction. Definitions, theorems, proofs, etc., are the invention of and exist only in the human mind. If you agree with me, great. If you disagree with me (e.g., you think proofs are discovered as opposed to created) so much the better! You can use the text in a point-counterpoint discussion. Ultimately, I do not think these philosophical questions have much bearing on the day-to-day work of mathematicians.

In any case, the essential point to convey from this section is that mathematical definitions must be much clearer than ordinary definitions. (Try having students define *chair* or *love*; as mathematicians we are lucky to be able to define our terms precisely.)

Furthermore, we cannot in this book define everything down to first principles. It is much too difficult for students on this level to go reduce everything to axiomatic set theory. It is reasonable for students to subsume sets and integers and to proceed from there.

3.1 (a) False. (b) True. (c) True. (d) True. (e) False. (f) False. (g) True. (h) True.

3.2 In the "official definition" (Definition 3.2) we have $0|0$, but in the alternative definition $0|0$ is false because $\frac{0}{0}$ is not an integer.

3.3 Only the first number, 21, is composite because $21 = 3 \times 7$ satisfies Definition 3.6. None of the other numbers are composite because they are not positive integers.

3.4 It is easiest if we define $\leq$ first. For integers x, y we say x is *less than or equal to* y, and we write $x \leq y$, provided $y - x$ is a natural number. We write $x < y$ provided $x \leq y$ and $x \neq y$. We write $x \geq y$ provided $y \leq x$. We write $x > y$ provided $y < x$.

3.5 Let x be an integer. Then x is also a rational number because we can write $x = \frac{x}{1}$, satisfying the definition of rational. On the other hand, $\frac{1}{2}$ is rational, but not an integer.

3.6 An integer x is called a *perfect square* provided there is an integer y such that $x = y^2$.

3.7 A number x is a *square root* of a number y provided $x^2 = y$.

3.8 The *perimeter* of a polygon is the sum of the lengths of its sides.

3.9 Suppose A, B, C are points in the plane. We say that C is *between* A and B provided the $d(A,C) + d(C,B) = d(A,B)$ where $d(\cdot,\cdot)$ denotes the distance between the two given points.

Suppose A, B, C are points in the plane. We say that they are collinear provided one of them is between the other two.

3.10 The *midpoint* of a line segment $\overline{AB}$ is a point C on the segment such that the distance from A to C equals the distance from C to B.

3.11 (a) A person X is called a *teenager* provided X is at least 13 years old and less than 20 years old.

(b) Person A is the *grandmother* of person B provided A is female and A is the parent of one of B's parents.

Alternatively: A person X is a *grandmother* if X is female and X is the parent of someone who is also a parent.

(c) A year is a *leap year* if it is 366 days long.

(d) A *dime* is a U.S. coin worth 10 cents, i.e., one-tenth of a dollar.

(e) A *palindrome* is a word that when written backwards spells the same word.

(f) A word X is called a *homophone* of a word Y if X and Y are spelled differently but are pronounced the same.

3.12 (a) 4. (b) 6. (c) $n+1$. (d) 4. (e) 9. (f) 49. (g) $(n+1)^2$. (h) 8. (i) 8. (j) $2^5 = 32$. (k) $8 \times 3 \times 2 \times 2 = 96$ because $8! = 2^7 \cdot 3^2 \cdot 5^1 \cdot 7^1$. (l) ∞.

The reason 30 and 42 have the same number of divisors is they are both the product of three distinct primes.

Prime factorization is developed formally later (see Section 39).

3.13 The first perfect number is 6. The next perfect number after 28 is 496. The divisors of 496 are 1, 2, 4, 8, 16, 31, 62, 124, and 248, and they sum to 496.

The following is a *Mathematica* program to find the next perfect number after 28.

```
IsPerfect[n_] :=
   Apply[Plus, Divisors[n]]==2*n;
n = 29;
While[Not[IsPerfect[n]], n++];
n
```

3.14 Presumably, the umpire's word is law. Regardless of what really happened or what the umpire saw, if the umpire says "Out!" by *definition*, the runner is out. This is akin to mathematical definitions. A number is *prime* because we crafted the definition a certain way.

4 Theorem

If Definition, Theorem, and Proof are our main characters, the most glamorous role is played by Theorem.

It is important for students to understand what a mathematical Statement is, and that Theorems are *true* mathematical statements that can be proved (with Proof to be introduced in the next section). Philosophical discussions are unavoidable here! We must distinguish *theorem* from the scientists' *theory* and be utterly rigid about what *truth* is. Compare and contrast statements such as

Love is wonderful versus *The square of an odd integer is odd*. The former is true, despite the fact that the words are undefined and sometimes love hurts or can be overbearing (and so isn't always wonderful). In mathematics, truth does not tolerate any exceptions.

In this section we discuss the various forms theorems can take (with if-then being the archetype) and synonyms for the word *theorem*. We also introduce *Mathspeak*. Mathematicians use the English language well, but our use of certain words is different from standard usage. Mathematicians are comfortable with these modifications to English, but they must be explicitly explained to newcomers to our discipline.

It may be tempting to skip the material on vacuous truth. However, I recommend you cover this. It is necessary for proving propositions such as for any set A, $\varnothing \subseteq A$.

New and noteworthy: We recommend you assign Exercise 4.12 to encourage students to create their own conjectures.

4.1 (a) If x is an odd integer and y is an even integer, then xy is even.

(b) If x is an odd integer, then x^2 is also odd.

(c) If p is a prime, then p^2 is not a prime.

(d) If x and y are integers with x and y negative, then xy is negative.

(e) If $ABCD$ is a rhombus, then $\overline{AC} \perp \overline{BD}$.

(f) If $\triangle ABC \cong \triangle XYZ$, then the area of $\triangle ABC$ equals the area of $\triangle XYZ$.

(g) If n is an integer, then $3|[n+(n+1)+(n+2)]$.

4.2 (a) If B, then A is true. The other statements are false.

(b) If A, then B is true. The other statements are false.

(c) If A, then B is true. The other statements are false.

(d) If A, then B is true. The other statements are false.

(e) If B, then A is true. The other statements are false.

(f) None of the three statements are true.

(g) If A, then B is true. The other statements are false.

(h) All three statements are true.

(i) All three statements are true.

(j) If B, then A is true. The other statements are false.

(k) If B, then A is true. The other statements are false.

4.3 Many answers are possible. Here is one. Let statement A be "the integer x is positive" and let B be the statement "the integer x^2 is positive."

The statement "If A, then B" is true, but the statement "If B, then A" is false.

4.4 Statement (a) is true unless A is true and B is false.

Statement (b) is true unless (not A) and B are both false, i.e., in case A is true and B is false.

Thus (a) and (b) are true under exactly the same situations, and therefore make the same assertion.

4.5 Statement (a) is true unless A is true and B is false.

Statement (b) is true unless (not B) is true and (not A) is false. This is the same as B is false and A is true.

Thus (a) and (b) are true under exactly the same situations, and therefore make the same assertion.

4.6 The statement "A iff B" is true precisely when A and B are both true, or both false.

The statement "(not A) iff (not B)" is true precisely when (not A) and (not B) are both true, or both false. This is the same as A and B are both false, or both true.

Thus both statements are true when A and B are both true, or both false; and both statements are false when A and B are not both true or both false.

The two statements are therefore true under the identical circumstances. In this sense, they make the same assertion.

4.7 The Pythagorean Theorem applies only to right triangles. Equilateral triangles are not right triangles.

4.8 The Pythagorean Theorem applies only to triangles in the plane. The "difficulty" comes from not being precise about the definition of *triangle*.

4.9 The sentence is nonsense because a line is a geometric object and distance is a number; these cannot be equal. Also, a line is an infinite object.

Here is a better way to write this: Of all paths (or curves) joining two given points A and B, the line segment with end points A and B has the smallest length.

4.10 This is vacuously true because guinea pigs don't have tails (or, at least, none to speak of).

4.11 *Lemmas* and *lemmata*.

4.12 (a) The sum of the first n odd numbers is n^2.

(b) The sum of the first n cubes is $\frac{1}{4}n^2(n+1)^2$. A very good partial answer is that the sum of the first n cubes is a perfect square.

(c) If n pairwise nonparallel lines are drawn in the plane, $(n^2+n+2)/2$ regions are created.

(d) When the path goes to every second point, we visit all points in case n is odd.

When the path goes to every third point, we visit all points in case n is not divisible by 3.

In the general case (skips of size k), we visit all points exactly when n and k have no common divisor greater than 1.

(e) The closed lockers are those whose numbers are perfect squares.

5 Proof

If the most glamorous part in the mathematical play is played by Theorem, the real star of the show is Proof. *Proof* is what distinguishes mathematics from all other disciplines. This section is an introduction to proof writing, but proof writing continues throughout the text.

Many forms of proof follow a rigid format. Scattered throughout the text are *proof templates* that are meant to demystify the proof-writing process.

One of the key steps to writing a proof is getting students simply to write the first and the last sentences of the proof. This sets up the questions: What do we know? and What do we want to show? Although these steps are obvious to working mathematicians, students are either reluctant and/or unable to do them. Their importance should be emphasized. Whenever I present a proof of a theorem in class, I insist that students give me the first and last sentences.

The next step is "unraveling definitions." Students should be taught to work both ends of the proof at once, unfolding the definitions in the hypothesis and working back from the conclusion. For simple problems, this is almost all that is needed. Once the appropriate facts are laid out in the proper form, bridging the gap from hypothesis to conclusion is easy (at least at this point in the course).

Only direct proofs of if-then and if-and-only-if statements are discussed at this point. More elaborate proof techniques follow in later sections.

Teaching proof writing in a vacuum is difficult. I use elementary number theory to demonstrate the methods. Be sure to emphasize to students that you know that *they know* that the sum of even integers is even. That is not the point in this section. The point is learning how to construct a proof, and that is easier to do if the proof deals with easy, familiar concepts.

Once you have done that simple example, you can prove the less obvious fact that n^3+1 is composite for integers $n \geq 2$.

5.1 Let x and y be odd integers. By definition, there are integers a and b such that $x = 2a+1$ and $y = 2b+1$. Therefore $x+y = (2a+1)+(2b+1) = 2a+2b+2 = 2(a+b+1)$. Therefore $x+y$ is divisible by 2. Therefore $x+y$ is even.

5.2 Let x be an odd integer and let y be an even integer. By definition, there is an integer a such that $x = 2a+1$. By definition, $2|y$, so (again, by definition) there is an integer b such that

$y = 2b$. We have $x + y = (2a+1) + (2b) = 2(a+b)+1$, and therefore, by definition, $x+y$ is odd.

5.3 Let n be an odd integer. Therefore there is an integer a such that $n = 2a+1$. Let b be the integer $-a-1$. Then

$$2b+1 = 2(-a-1)+1 = -2a-2+1 = -2a-1 = -(2a+1) = -n.$$

Since $-n = 2b+1$ where b is an integer, $-n$ is odd.

5.4 Let x and y be even integers. Therefore $x = 2a$ and $y = 2b$ for some integers a and b. Now $xy = (2a)(2b) = 4ab = 2(2ab)$ and so xy is even.

5.5 Let x be an even integer and let y be an odd integer. Therefore there are integers a and b with $x = 2a$ and $y = 2b+1$. Note that $xy = (2a)(2b+1) = 2(2ab+a)$ and so xy is even.

5.6 Let x, y be odd. There exist integers a, b such that $x = 2a+1$ and $y = 2b+1$ and so $xy = (2a+1)(2b+1) = 4ab+2a+2b+1 = 2(2ab+a+b)+1$ and so xy is odd.

5.7 This is a corollary of Exercise 5.6.

5.8 *Proof #1*: Let n be an odd integer. Thus $n = 2a+1$ for some integer a. Therefore

$$n^3 = (2a+1)^3 = 8a^3+12a^2+6a+1 = 2(4a^3+6a^2+3a)+1 = 2b+1$$

where $b = 4a^3+6a^2+3a$ is an integer. Since $n^3 = 2b+1$, n^3 is odd. □

Proof #2: Let n be an odd integer. By Exercise 5.6, $n \cdot n = n^2$ is odd. And since n^2 and n are odd, again by Exercise 5.6, $n^2 \cdot n = n^3$ is odd. □

5.9 Suppose $a|b$ and $a|c$. By definition, there are integers x and y with $b = ax$ and $c = ay$. Therefore $b+c = ax+ay = a(x+y)$ and so $b+c$ is divisible by a, i.e., $a|(b+c)$.

5.10 Suppose $a|b$. Then there is an integer x such that $ax = b$. Multiplying both sides by c we have $a(xc) = bc$ and so $a|(bc)$.

5.11 We are given that $d|a$ and $d|b$. By the previous problem, $d|(ax)$ and $d|(by)$. By the problem before that, $d|[(ax)+(by)]$.

5.12 Suppose $a|b$ and $c|d$. By definition, there are integers x and y with $ax = b$ and $cy = d$. Multiplying these equations we get $bd = (ax)(cy) = (ac)(xy)$ and so bd is divisible by ac.

5.13 ($\Rightarrow$) Suppose x is odd. Therefore $x = 2a+1$ for some integer a. Now $x+1 = (2a+1)+1 = 2a+2 = 2(a+1)$, and so $2|(x+1)$ and therefore $x+1$ is even.

($\Leftarrow$) Suppose $x+1$ is even. Therefore $2|(x+1)$ and hence there is an integer b such that $x+1 = 2b$. Thus $x = 2b-1 = 2(b-1)+1$ and so x is odd.

5.14 ($\Rightarrow$) Suppose x is odd. Then there is an integer a such that $x = 2a+1$. Let $b = a+1$; notice that b is an integer. Observe that

$$2b-1 = 2(a+1)-1 = 2a+2-1 = 2a+1 = x.$$

Therefore $x = 2b-1$ for some integer b.

($\Leftarrow$) Suppose $x = 2b-1$ for some integer b. Let $a = b-1$; note that a is an integer. Observe that

$$2a+1 = 2(b-1)+1 = 2b-2+1 = 2b-1 = x.$$

Since $x = 2a+1$ for an integer a, it follows that x is odd.

5.15 ($\Rightarrow$) Suppose $0|x$. This means there is an integer a so that $0 \cdot a = x$. Therefore $x = 0$.

($\Leftarrow$) Clearly $0|0$ because $0 \cdot 1 = 0$.

5.16 ($\Rightarrow$) Suppose $a < b$. Therefore $b-a > 0$, so $b-a$ is a positive integer. Since 1 is the smallest positive integer, $b-a \geq 1$ whence $b-1 \geq a$, in other words, $a \leq b-1$.

($\Leftarrow$) Suppose $a \leq b-1$. Since $b-1 < b$ we have $a < b$.

5.17 ($\Rightarrow$) Suppose that x is strictly between 1 and $\sqrt{a}$; that is, $1 < x < \sqrt{a}$. Since $1 < x$ we have that $1/1 > 1/x$; multiplying both sides by a gives $a > a/x$. Likewise, since $x < \sqrt{a}$ we have $1/x > 1/\sqrt{a}$; multiplying both sides by a gives $a/x > a/\sqrt{a} = \sqrt{a}$. Thus $\sqrt{a} < a/x < a$; that is, a/x is strictly between $\sqrt{a}$ and a.

($\Leftarrow$) Suppose that x is strictly between $\sqrt{a}$ and a; that is, $\sqrt{a} < x < a$. Since $\sqrt{a} < x$, we have that $1/\sqrt{a} > 1/x$. Multiplying both sides by a gives $\sqrt{a} = a/\sqrt{a} > a/x$. Similarly, since $x < a$ we have that $1/x > 1/a$. Multiplying through by a gives $a/x > 1$. Thus $1 < a/x < \sqrt{a}$ and thus we see that a/x is strictly between 1 and $\sqrt{a}$.

5.18 Let n^2 and $(n+1)^2$ be consecutive perfect squares (where n is an integer). Their difference is $(n+1)^2 - n^2 = (n^2+2n+1) - n^2 = 2n+1$ which is odd by Definition 3.4. □

5.19 Suppose a is a perfect square. Then there is some integer n so that $n^2 = a$. If $n \geq 0$ there is nothing more to prove. But if $n < 0$, then $-n > 0$ and note that $(-n)^2 = n^2 = a$, so a is the square of a positive integer. □

5.20 We are given that $0 < a < b$. Since a is positive, multiplying both sides of $a < b$ by a gives $a^2 < ab$. Likewise, since b is positive, multiplying both sides of $a < b$ by b gives $ab < b^2$. Since $a^2 < ab$ and $ab < b^2$, we have $a^2 < b^2$. □

5.21 Suppose the perfect squares are A and B where $A = a^2$ and $B = b^2$ for integers a and b. By Exercise 5.19 we may assume a and b are nonnegative.

Since A and B are distinct nonconsecutive squares, we know that a and b differ by at least 2, say $b \geq a+2$.

Thus $B-A = b^2-a^2 = (b-a)(b+a)$. Since $b \geq a+2$ and $a \geq 0$, we know $b \geq 2$ and so $b+a \geq 2$. From $b \geq a+2$ we also know (by subtracting a from both sides) that $b-a \geq 2$. Thus $B-A$ is the product of two integers, both of which are at least 2. Therefore $B-A$ is composite. □

5.22 (⇒) Suppose x is odd. By definition, there is an integer a such that $x = 2a+1 = a+(a+1)$ and so x is the sum of two consecutive integers.

(⇐) Suppose x is the sum of two consecutive integers, say b and $b+1$. Then $x = b+(b+1) = 2b+1$, and so x is odd.

5.23 Consider a statement such as, "Let $x \geq 3$ be an integer. If x is even or x is square, then x is not prime." If we only prove "If x is even, then x is not prime" then we have not covered the case when x is square but not even (e.g., $x = 25$).

If we have shown $A \Rightarrow C$ and $B \Rightarrow C$, then we have covered all the possibilities.

Worse, if we only prove $A \Rightarrow C$, we have have a "proof" of an invalid statement. Consider the statement, "Let x be an integer with $x \geq 3$. If x is even or if x is odd, then x is not prime." Of course, this statement is false, so proving only "If x is even, then x is not prime" is not a valid proof.

This problem is easier, and I recommend it be reassigned, after Section 7. Students can then make a truth table of $(a \vee b) \rightarrow c$ and $(a \rightarrow c) \wedge (b \rightarrow c)$.

This is the point of Exercise 7.8.

5.24 Suppose we proved $A \Rightarrow B$ and $(\text{not } A) \Rightarrow (\text{not } B)$.

Then when A is true, we must have B be true (by the first proof).

Now suppose B is true, so $(\text{not } B)$ is false. By the second proof, this can only happen if $(\text{not } A)$ is false, i.e., A is true.

Thus A iff B is proven.

As with the previous problem, this is more natural after we do Section 7. See Exercise 7.7.

6 Counterexample

A counterexample is a form of proof; it proves that a statement is false. Counterexamples are conceptually simpler than proofs. However, students sometimes have difficulty understanding what constitutes a counterexample.

Be sure to emphasize: For an if-then statement, any example that satisfies the *if* but not the *then* will do. For an if-and-only-if statement, a counterexample to either implication suffices.

There is an urban legend about a student who was presented with a test item that asked students to prove or disprove a certain statement. The statement was false, so a counterexample was appropriate. The student wrote an erroneous proof. When the question was marked wrong, the student sought out the professor. The professor explained that the statement is false and showed the student a counterexample. "That's fine," said the student, "but the problem says to *prove or disprove*. You chose to disprove, but I decided to prove!"

We need to be sure our students understand the nature of mathematical truth and the role of proof and counterexample!

6.1 There are many possible counterexamples. Let $a = 3$ and $b = -6$. Then $a|b$ but $a \not\leq b$.

6.2 Let $a = 10$ and $b = 0$. Then $a|b$ but $a \not\leq b$.

6.3 For example, let $a = 15$, $b = 3$, and $c = 5$. Then $a|(bc)$ but a divides neither b nor c.

6.4 Let $a = 2$, $b = 3$, and $c = 4$. Note that $a^{(b^c)} = 2^{3^4} = 2^{81}$ but $(a^b)^c = (2^3)^4 = 2^{12}$.

6.5 Counterexample: $p = 2$ and $q = 5$ are primes, but $p + q = 7$ is not composite.

6.6 11 is prime, but $2^{11} - 1 = 2047 = 23 \times 89$ is not.

6.7 $2^{2^5} + 1 = 641 \times 6700417$.

6.8 131 is a palindrome, but it is not divisible by 11.

6.9 (a) For $n = 1, 2, \ldots, 10$ the values of $n^2 + n + 41$ are, respectively, 43, 47, 53, 61, 71, 83, 97, 113, 131, and 151, all of which are prime.

(b) However, for $n = 41$, clearly $n^2 + n + 41$ is divisible by 41. (Indeed $41^2 + 41 + 41 = 1763 = 43 \cdot 41$.)

6.10 A counterexample to the statement "*A* iff *B*" can be either an instance where *A* is true and *B* is false, or an instance where *B* is true and *A* is false.

6.11 $x = 0$ is a counterexample; the only counterexample.

6.12 Let the first right triangle have side lengths 6, 8, and 10, and let the second right triangle have side lengths $\sqrt{50}$, $\sqrt{50}$, and 10. The areas of these triangles are 24 and 25, respectively.

6.13 The positive integer 9 is composite, but does not have two different prime factors.

7 Boolean Algebra

It is natural to introduce symbolic logic together with proof writing. Furthermore, it is useful to have a mechanical way to deal with logical reasoning. Students new to logical thinking can become confused between a statement and its converse, and can find it difficult to understand why a statement and its contrapositive are logically equivalent. Boolean algebra makes these issues more mechanical.

The notation of Boolean algebra is useful in mathematics and computer science.

The downside to Boolean algebra is that it can be boring. My recommendation is to move quickly through this subject.

Truth tables are easy for students to grasp, and provide an easy way to prove results. It is possible to prove logical equivalences using Theorem 7.2, but this can be time consuming, and there are more interesting topics ahead.

Exercises 7.11(g) and (h) foreshadow the technique of proof by contradiction, so we recommend that they be assigned.

So the bottom line is: The material in this section is worthwhile, but should not be overemphasized. However, there are two interesting projects you can assign based on this material. For electrical engineers, you can have your students create circuits (either using discrete components or in an emulator program such as Spice) to model Boolean expressions. For computer scientists, you can assign a project in which students write a program to evaluate Boolean expressions, test if those expressions are tautologies, test if two expressions are logically equivalent, etc.

7.1 (a) FALSE. (b) TRUE. (c) FALSE. (d) FALSE. (e) TRUE.

The answer to (a) does not depend on how the expression is parenthesized.

7.2 Truth tables for the various parts of Theorem 7.2.

$x \wedge y = y \wedge x$			
x	y	$x \wedge y$	$y \wedge x$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

$x \vee y = y \vee x$			
x	y	$x \vee y$	$y \vee x$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

$(x \wedge y) \wedge z = x \wedge (y \wedge z)$						
x	y	z	$x \wedge y$	$(x \wedge y) \wedge z$	$y \wedge z$	$x \wedge (y \wedge z)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	F	F	T	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

$(x \vee y) \vee z = x \vee (y \vee z)$						
x	y	z	$x \vee y$	$(x \vee y) \vee z$	$y \vee z$	$x \vee (y \vee z)$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	T	T	F	T
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	F	T	T	T
F	F	F	F	F	F	F

$x = x \wedge \mathsf{T}$	
x	$x \wedge \mathsf{T}$
T	T
F	F

$x = x \vee \mathsf{F}$	
x	$x \vee \mathsf{F}$
T	T
F	F

$x = \neg(\neg x)$		
x	$\neg x$	$\neg(\neg x)$
T	F	T
F	T	F

$x = x \wedge x$	
x	$x \wedge x$
T	T
F	F

$x = x \vee x$	
x	$x \vee x$
T	T
F	F

$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$							
x	y	z	$y \vee z$	$x \wedge (y \vee z)$	$x \wedge y$	$x \wedge z$	$(x \wedge y) \vee (x \wedge z)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$							
x	y	z	$y \wedge z$	$x \vee (y \wedge z)$	$x \vee y$	$x \vee z$	$(x \vee y) \wedge (x \vee z)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

$x \wedge (\neg x) =$ FALSE		
x	$\neg x$	$x \wedge (\neg x)$
T	F	F
F	T	F

$x \vee (\neg x) =$ TRUE		
x	$\neg x$	$x \vee (\neg x)$
T	F	T
F	T	T

$\neg(x \wedge y) = (\neg x) \vee (\neg y)$						
x	y	$x \wedge y$	$\neg(x \wedge y)$	$\neg x$	$\neg y$	$(\neg x) \vee (\neg y)$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

$\neg(x \vee y) = (\neg x) \wedge (\neg y)$						
x	y	$x \vee y$	$\neg(x \vee y)$	$\neg x$	$\neg y$	$(\neg x) \wedge (\neg y)$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

7.3 We make a truth table for $(x \wedge y) \vee (x \wedge \neg y)$ and compare to x.

x	y	$x \wedge y$	$x \wedge \neg y$	$(x \wedge y) \vee (x \wedge \neg y)$	x
T	T	T	F	T	T
T	F	F	T	T	T
F	T	F	F	F	F
F	F	F	F	F	F

7.4 Truth table for $x \rightarrow y$ and $(\neg y) \rightarrow (\neg x)$.

x	y	$x \rightarrow y$	$\neg y$	$\neg x$	$(\neg y) \rightarrow (\neg x)$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

7.5 Truth table for $x \leftrightarrow y$ and $(\neg x) \leftrightarrow (\neg y)$.

x	y	$x \leftrightarrow y$	$\neg x$	$\neg y$	$(\neg x) \leftrightarrow (\neg y)$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

7.6 Truth table for $x \leftrightarrow y$ and $(x \rightarrow y) \wedge (y \rightarrow x)$.

x	y	$x \leftrightarrow y$	$x \rightarrow y$	$y \rightarrow x$	$(x \rightarrow y) \wedge (y \rightarrow x)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

7.7 Truth table for $x \leftrightarrow y$ and $(x \rightarrow y) \wedge ((\neg x) \rightarrow (\neg y))$.

x	y	$x \leftrightarrow y$	$x \to y$	$(\neg x) \to (\neg y)$	$(x \to y) \wedge$ $(\neg x) \to (\neg y)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

7.8 Truth table for $(x \vee y) \to z$ and $(x \to z) \wedge (y \to z)$.

x	y	z	$x \vee$ y	$(x \vee y)$ $\to z$	$x \to$ z	$y \to$ z	$(x \to z) \wedge$ $(y \to z)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	T	T
T	F	F	T	F	F	T	F
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	F
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

7.9 $2^{10} = 1024$. Each addition of a variable doubles the number of rows.

7.10 To disprove a logical equivalence between two formulas, find values for the variables that give *different* values for the formulas.

Correct answers to these include: (a) $x = \text{TRUE}, y = \text{FALSE}$, (b) $x = \text{FALSE}, y = \text{TRUE}$, and (c) $x = \text{TRUE}, y = \text{TRUE}$. Other answers may be possible.

7.11 The last column of the truth table should contain the entry TRUE in every row.

(a)

x	y	$x \vee y$	$x \vee \neg y$	$(x \vee y) \vee$ $(x \vee \neg y)$
T	T	T	T	T
T	F	T	T	T
F	T	T	F	T
F	F	F	T	T

(b)

x	y	$x \to y$	$x \wedge$ $(x \to y)$	$(x \wedge (x \to y))$ $\to y$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

(c)

x	$\neg x$	$\neg(\neg x)$	$x \leftrightarrow \neg(\neg x)$
T	F	T	T
F	T	F	T

(d)

x	$x \to x$
T	T
F	T

(e)

x	y	z	$x \to y$	$y \to z$	$(x \to y) \wedge (y \to z)$	$x \to z$	$((x \to y) \wedge (y \to z)) \to (x \to z)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

(f)

x	$\mathsf{F} \to x$
T	T
F	T

(g)

x	$x \to \mathsf{F}$	$\neg x$	$(x \to \mathsf{F}) \to \neg x$
T	F	F	T
F	T	T	T

(h)

x	y	$x \to y$	$x \to \neg y$	$(x \to y) \wedge (x \to \neg y)$	$\neg x$	$((x \to y) \wedge (x \to \neg y)) \to \neg x$
T	T	T	F	F	F	T
T	F	F	T	F	F	T
F	T	T	T	T	T	T
F	F	T	T	T	T	T

7.12 A solution to (a) is given in the Hints and a solution to (b) in the statement of the problem. The other formulas are proved here.

(c) $(\neg(\neg x)) \leftrightarrow x = x \leftrightarrow x = (x \to x) \wedge (x \to x) = (x \to x) = \neg x \vee x = \mathsf{T}$.

(d) $x \to x = (x \to x) \wedge (x \to x) = (x \to x) = \neg x \vee x = \mathsf{T}$.

(e) This one qualifies as cruel and unusual punishment:

$$
\begin{aligned}
((x \to y) \wedge (y \to z)) \to (x \to z) &= [(\neg x \vee y) \wedge (\neg y \vee z)] \to (x \to z) \\
&= [(\neg x \wedge (\neg y \vee z)) \vee (y \wedge (\neg y \vee z))] \to (x \to z) \\
&= [(\neg x \wedge \neg y) \vee (\neg x \wedge z) \vee (y \wedge \neg y) \vee (y \wedge z)] \to (x \to z) \\
&= [(\neg x \wedge \neg y) \vee (\neg x \wedge z) \vee \mathsf{F} \vee (y \wedge z)] \to (x \to z) \\
&= [(\neg x \wedge \neg y) \vee (\neg x \wedge z) \vee (y \wedge z)] \to (x \to z) \\
&= \neg[(\neg x \wedge \neg y) \vee (\neg x \wedge z) \vee (y \wedge z)] \vee (x \to z) \\
&= \neg[(\neg x \wedge \neg y) \vee (\neg x \wedge z) \vee (y \wedge z)] \vee \neg x \vee z \\
&= \neg[(\neg x \wedge \neg y) \vee (\neg x \wedge z) \vee (y \wedge z)] \vee \neg x \vee \neg\neg z \\
&= \neg\Big[[(\neg x \wedge \neg y) \vee (\neg x \wedge z) \vee (y \wedge z)] \wedge x \wedge \neg z\Big] \\
&= \neg\Big[[((\neg x \wedge \neg y) \wedge x) \vee ((\neg x \wedge z) \wedge x) \vee ((y \wedge z) \wedge x)] \wedge \neg z\Big] \\
&= \neg\Big[[\mathsf{F} \vee \mathsf{F} \vee \mathsf{F} \vee ((y \wedge z) \wedge x)] \wedge \neg z\Big] \\
&= \neg[(y \wedge z \wedge x) \wedge \neg z] = \neg\mathsf{F} = \mathsf{T}.
\end{aligned}
$$

(f) $\mathsf{F} \to x = \neg\mathsf{F} \vee x = \mathsf{T} \vee x = \mathsf{T}$.

(g) $(x \to \mathsf{F}) \to \neg x = (\neg x \vee \mathsf{F}) \to \neg x = \neg x \to \neg x = \neg\neg x \vee \neg x = x \vee \neg x = \mathsf{T}$.

(h)

$$
\begin{aligned}
((x \to y) \wedge (x \to \neg y)) \to \neg x &= [(\neg x \vee y) \wedge (\neg x \vee \neg y) \to \neg x \\
&= [\neg x \vee (y \wedge \neg y)] \to \neg x \\
&= [\neg x \vee \mathsf{F}] \to \neg x \\
&= \neg x \to \neg x \\
&= \neg\neg x \vee \neg x = x \vee \neg x = \mathsf{T}.
\end{aligned}
$$

7.13 (a)

x	y	$x \vee y$	$x \vee \neg y$	$\neg x$	$(x \vee y) \wedge (x \vee \neg y) \wedge \neg x$
T	T	T	T	F	F
T	F	T	T	F	F
F	T	T	F	T	F
F	F	F	T	T	F

(b)

x	y	$x \to y$	$\neg y$	$x \wedge (x \to y) \wedge (\neg y)$
T	T	T	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	F

(c)

x	y	$x \to y$	$(\neg x) \to y$	$(x \to y) \wedge ((\neg x) \to y) \wedge \neg y$
T	T	T	T	F
T	F	F	T	F
F	T	T	T	F
F	F	T	F	F

7.14 ($\Rightarrow$) Suppose A is logically equivalent to B. This means that whenever we substitute for the variables in A and B we get the same truth value. Since $\mathsf{T} \leftrightarrow \mathsf{T} = \mathsf{T}$ and $\mathsf{F} \leftrightarrow \mathsf{F} = \mathsf{T}$, whenever we substitute into $A \leftrightarrow B$ we always get T. Therefore, $A \leftrightarrow B$ is a tautology.

($\Leftarrow$) Suppose $A \leftrightarrow B$ is a tautology. Then whenever we substitute into the variables in A and B we get $A \leftrightarrow B = \text{TRUE}$. Now the only way that $\leftrightarrow$ evaluates to TRUE is when both A and B have the same truth value (because $\mathsf{T} \leftrightarrow \mathsf{F} = \mathsf{F} \leftrightarrow \mathsf{T} = \mathsf{F}$) and so A and B always yield the same truth value upon substitution for their variables. Thus A is logically equivalent to B. □

7.15 The expression $x \leftrightarrow y$ is logically equivalent to $(x \wedge y) \vee ((\neg x) \wedge (\neg y))$. We can show this via truth table:

x	y	$x \leftrightarrow y$	$x \wedge y$	$(\neg x) \wedge (\neg y)$	$(x \wedge y) \vee ((\neg x) \wedge (\neg y))$
T	T	T	T	F	T
T	F	F	F	F	F
F	T	F	F	F	F
F	F	T	F	T	T

Other answers are possible.

7.16 For (a) it is easy to see that $\veebar$ is commutative as $\mathsf{F} \veebar \mathsf{T} = \mathsf{T} \veebar \mathsf{F} = \mathsf{T}$. To show that $\veebar$ is associative, we can use a truth table:

x	y	z	$x \veebar y$	$(x \veebar y) \veebar z$	$y \veebar z$	$x \veebar (y \veebar z)$
T	T	T	F	T	F	T
T	T	F	F	F	T	F
T	F	T	T	F	T	F
T	F	F	T	T	F	T
F	T	T	T	F	F	F
F	T	F	T	T	T	T
F	F	T	F	T	T	T
F	F	F	F	F	F	F

For (b) we show $x \veebar y = (x \wedge \neg y) \vee ((\neg x) \wedge y)$ via the following truth table:

x	y	$x \veebar y$	$x \wedge \neg y$	$(\neg x) \wedge y$	$(x \wedge \neg y) \vee ((\neg x) \wedge y)$
T	T	F	F	F	F
T	F	T	T	F	T
F	T	T	F	T	T
F	F	F	F	F	F

For (c) we show $x \veebar y = (x \vee y) \wedge (\neg(x \wedge y))$ via the following truth table.

x	y	$x \veebar y$	$x \vee y$	$\neg(x \wedge y)$	$(x \vee y) \wedge (\neg(x \wedge y))$
T	T	F	T	F	F
T	F	T	T	T	T
F	T	T	T	T	T
F	F	F	F	T	F

We call $\veebar$ *exclusive* or because it captures the exclusive nature of English *or*: you may have option *A* or *B*, *but not both*. In the same way $x \veebar y$ when x or y *but not both* are TRUE.

7.17 There are $2^4 = 16$ possible binary Boolean operations.

7.18 Consider the four expressions

$$\begin{array}{rr} x \wedge y & x \wedge (\neg y) \\ (\neg x) \wedge y & (\neg x) \wedge (\neg y) \end{array}$$

Notice that in each case there is a *unique* substitution for x and y that makes the expression TRUE. For example, $(\neg x) \wedge y$ is TRUE exactly when $x =$ FALSE and $y =$ TRUE (and otherwise it is FALSE).

We can combine these basic four expressions using $\vee$s to make any binary Boolean operation we want. We just decide which of the ? entries in the chart

x	y	$x * y$
TRUE	TRUE	?
TRUE	FALSE	?
FALSE	TRUE	?
FALSE	FALSE	?

we want to be TRUE and we link together the appropriate expressions above with $\vee$s. For example, if we want

x	y	$x * y$
TRUE	TRUE	TRUE
TRUE	FALSE	FALSE
FALSE	TRUE	TRUE
FALSE	FALSE	FALSE

we would take $x \wedge y$ together with $(\neg x) \wedge y$ as

$$x * y = (x \wedge y) \vee ((\neg x) \wedge y).$$

For the special case where all entries in the last column are TRUE, we can use $x * y = x \vee \neg x$, and in case all the entries in the last column are FALSE, we can use $x * y = x \wedge \neg x$.

Therefore all 16 binary Boolean functions can be expressed in terms of $\wedge$, $\vee$, and $\neg$.

7.19 Note that $x \vee y = \neg((\neg x) \wedge (\neg y))$ and so in any expression that uses $\vee$ we can replace $\vee$ with $\wedge$s and $\neg$s.

7.20 For (a), a truth table for $\overline{\wedge}$:

x	y	$x \,\overline{\wedge}\, y$
T	T	F
T	F	T
F	T	T
F	F	T

For (b), the operation $\overline{\wedge}$ is commutative but not associative. Commutativity is clear from the defining truth table and from the fact that $x \,\overline{\wedge}\, y = \neg(x \wedge y)$:

$$x \,\overline{\wedge}\, y = \neg(x \wedge y) = \neg(y \wedge x) = y \,\overline{\wedge}\, x.$$

However, to see that $\overline{\wedge}$ is not associative, we compute:

$$\mathsf{T} \,\overline{\wedge}\, (\mathsf{T} \,\overline{\wedge}\, \mathsf{F}) = \mathsf{T} \,\overline{\wedge}\, \mathsf{T} = \mathsf{F} \qquad \text{and} \qquad (\mathsf{T} \,\overline{\wedge}\, \mathsf{T}) \,\overline{\wedge}\, \mathsf{F} = \mathsf{F} \,\overline{\wedge}\, \mathsf{F} = \mathsf{T}.$$

For (c), we have $\neg x = \neg(x \wedge x) = x \,\overline{\wedge}\, x$, and since $x \wedge y = \neg(x \,\overline{\wedge}\, y)$ we have

$$x \wedge y = (x \,\overline{\wedge}\, y) \,\overline{\wedge}\, (x \,\overline{\wedge}\, y).$$

Therefore (for (d)) we have that $\neg$ and $\wedge$ can be expressed in terms of $\overline{\wedge}$. Since all Boolean operations can be expressed just in terms of $\neg$ and $\wedge$, we can express all binary Boolean operations just in terms of $\overline{\wedge}$.