Solutions to Exercises in Chapter 1

1.6.1 Check that the formula $A = \frac{1}{4}(a + c)(b + d)$ works for rectangles but not for parallelograms.

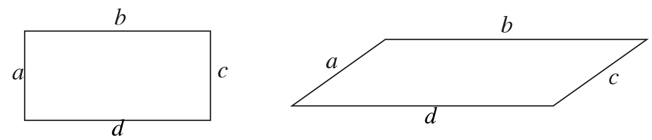

FIGURE S1.1: Exercise 1.6.1. A rectangle and a parallelogram

For rectangles and parallelograms, $a = c$ and $b = d$ and $Area = base * height$.
For a rectangle, the base and the height will be equal to the lengths of two adjacent sides. Therefore $A = a * d = \frac{1}{2}(a + a) * \frac{1}{2}(b + b) = \frac{1}{4}(a + c)(b + d)$
In the case of a parallelogram, the height is smaller than the length of the side so the formula does not give the correct answer.

1.6.2 The area of a circle is given by the formula $A = \pi(\frac{d}{2})^2$. According the Egyptians, A is also equal to the area of a square with sides equal to $\frac{8}{9}d$; thus $A = (\frac{8}{9})^2 d^2$. Equating and solving for π gives

$$\pi = \frac{(\frac{8}{9})^2 d^2}{\frac{1}{4}d^2} = \frac{\frac{64}{81}}{\frac{1}{4}} = \frac{256}{81} \approx 3.160494.$$

1.6.3 The sum of the measures of the two acute angles in $\triangle ABC$ is 90°, so the first shaded region is a square. We must show that the area of the shaded region in the first square (c^2) is equal to the area of the shaded region in the second square $(a^2 + b^2)$. The two large squares have the same area because they both have side length $a + b$. Also each of these squares contains four copies of triangle $\triangle ABC$ (in white). Therefore, by subtraction, the shadesd regions must have equal area and so $a^2 + b^2 = c^2$.

1.6.4 **(a)** Suppose $a = u^2 - v^2$, $b = 2uv$ and $c = u^2 + v^2$. We must show that $a^2 + b^2 = c^2$. First, $a^2 + b^2 = (u^2 - v^2)^2 + (2uv)^2 = u^4 - 2u^2v^2 + v^4 + 4u^2v^2 = u^4 + 2u^2v^2 + v^4$ and, second, $c^2 = (u^2 + v^2)^2 = u^4 + 2u^2v^2 + v^4 = u^4 + 2u^2v^2 + v^4$. So $a^2 + b^2 = c^2$.

(b) Let u and v be odd. We will show that a, b and c are all even. Since u and v are both odd, we know that u^2 and v^2 are also odd. Therefore $a = u^2 - v^2$ is even (the difference between two odd numbers is even). It is obvious that $b = 2uv$ is even, and $c = u^2 + v^2$ is also even since it is the sum of two odd numbers.

(c) Suppose one of u and v is even and the other is odd. We will show that a, b, and c do not have any common prime factors. Now a and c are both odd, so 2 is not a factor of a or c. Suppose $x \neq 2$ is a prime factor of b. Then either x divides u or x divides v, but not both because u and v are relatively prime. If x divides u, then it also divides u^2 but not v^2. Thus x is not a factor of a or c.

If x divides v, then it divides v^2 but not u^2. Again x is not a factor of a or c. Therefore (a, b, c) is a primitive Pythagorean triple.

1.6.5 Let $h + x$ be the height of the entire (untruncated) pyramid. We know that

$$\frac{h+x}{x} = \frac{a}{b}$$

(by the Similar Triangles Theorem), so $x = h\frac{b}{a-b}$ (algebra). The volume of the truncated pyramid is the volume of the whole pyramid minus the volume of the top pyramid. Therefore

$$\begin{aligned} V &= \frac{1}{3}(h+x)a^2 - \frac{1}{3}xb^2 \\ &= \frac{1}{3}(h + h\frac{b}{a-b})a^2 - \frac{1}{3}h(\frac{b^3}{a-b}) \\ &= \frac{h}{3}(a^2 + \frac{a^2b}{a-b}) - \frac{h}{3}(\frac{b^3}{a-b}) \\ &= \frac{h}{3}(a^2 + \frac{a^2b - b^3}{a-b}) \\ &= \frac{h}{3}(a^2 + \frac{(a-b)(ab+b^2)}{(a-b)}) \\ &= \frac{h}{3}(a^2 + ab + b^2). \end{aligned}$$

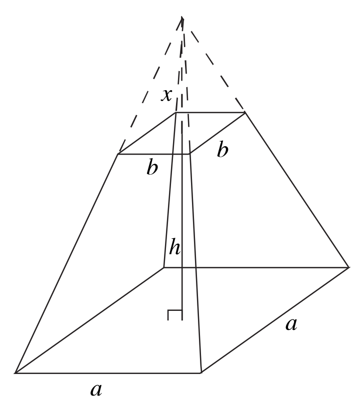

FIGURE S1.2: Exercise 1.6.5. A truncated pyramid.

1.6.6 Constructions using a compass and a straightedge. There are numerous ways in which to accomplish each of these constructions; just one is indicated in each case.

(a) The perpendicular bisector of a line segment $\overline{AB}$.
Using the compass, construct two circles, the first about A through B, the second about B through A. Then use the straightedge to construct a line through the two points created by the intersection of the two circles.

(b) A line through a point P perpendicular to a line ℓ.
Use the compass to construct a circle about P, making sure the circle is big enough so that it intersects ℓ at two points, A and B. Then construct the perpendicular bisector of segment $\overline{AB}$ as in part (a).

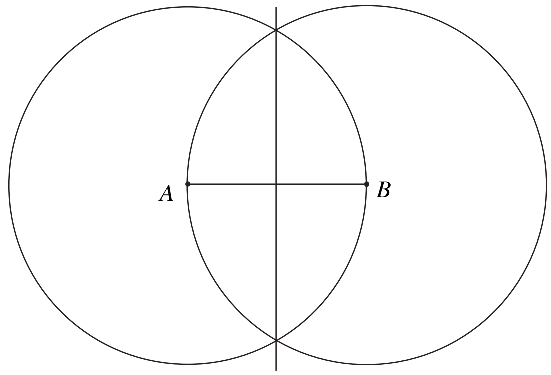

FIGURE S1.3: Exercise 1.6(a) Construction of a perpendicular bisector

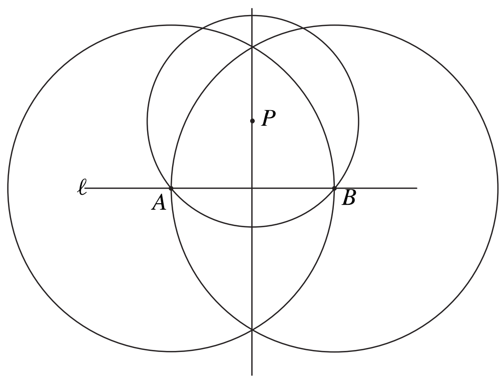

FIGURE S1.4: Exercise 1.6.6(b) Construction of a line through P, perpendicular to ℓ

(c) The angle bisector of $\angle BAC$.
Using the compass, construct a circle about A that intersects $\overline{AB}$ and $\overline{AC}$. Call those points of intersection D and E respectively. Then construct the perpendicular bisector of $\overline{DE}$. This line is the angle bisector.

1.6.7 **(a)** No. Euclid's postulates say nothing about the number of points on a line.
(b) No.
(c) No. The postulates only assert that there is a line; they do not say there is only one.

1.6.8 The proof of Proposition 29.

1.6.9 Let $\square ABCD$ be a rhombus (all four sides are equal), and let E be the point of intersection between $\overline{AC}$ and $\overline{DB}$.[1] We must show that $\triangle AEB \cong \triangle CEB \cong \triangle CED \cong \triangle AED$. Now $\angle BAC \cong \angle ACB$ and $\angle CAD \cong \angle ACD$ by Proposition 5. By addition we can see that $\angle BAD \cong \angle BCD$ and similarly, $\angle ADC \cong \angle ABC$. Now we know that $\triangle ABC \cong \triangle ADC$ by Proposition 4. Similarly, $\triangle DBA \cong \triangle DBC$. This implies that $\angle BAC \cong \angle DAC \cong \angle BCA \cong \angle DCA$ and $\angle BDA \cong \angle DBA \cong \angle BDC \cong \angle DBC$.

[1] In this solution and the next, the existence of the point E is taken for granted. Its existence is obvious from the diagram. Proving that E exists is one of the gaps that must be filled in these proofs. This point will be addressed in Chapter 6.

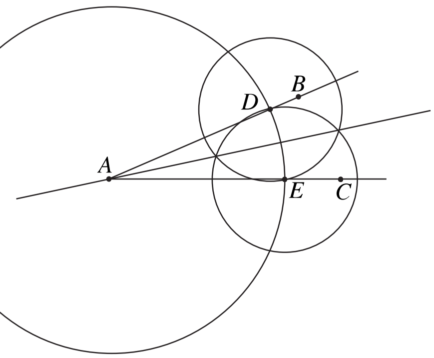

FIGURE S1.5: Exercise 1.6.6(c) Construction of an angle bisector

Thus $\triangle AEB \cong \triangle AED \cong \triangle CEB \cong \triangle CED$, again by Proposition 4.[2]

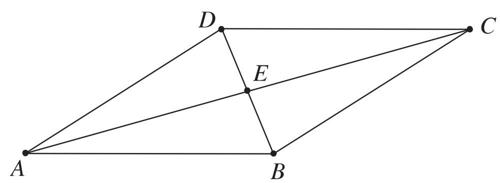

FIGURE S1.6: Exercise 1.6.9 Rhombus $\square ABCD$

1.6.10 Let $\square ABCD$ be a rectangle, and let E be the point of intersection of $\overline{AC}$ and $\overline{BD}$. We must prove that $\overline{AC} \cong \overline{BD}$ and that $\overline{AC}$ and $\overline{BD}$ bisect each other (i.e., $\overline{AE} \cong \overline{EC}$ and $\overline{BE} \cong \overline{ED}$). By Proposition 28, $\overleftrightarrow{DA} \parallel \overleftrightarrow{CB}$ and $\overleftrightarrow{DC} \parallel \overleftrightarrow{AB}$. Therefore, by Proposition 29, $\angle CAB \cong \angle ACD$ and $\angle DAC \cong \angle ACB$. Hence $\triangle ABC \cong \triangle CDA$ and $\triangle ADB \cong \triangle CBD$ by Proposition 26 (ASA). Since those triangles are congruent we know that opposite sides of the rectangle are congruent and $\triangle ABD \cong \triangle BAC$ (by Proposition 4), and therefore $\overline{BD} \cong \overline{AC}$.

Now we must prove that the segments bisect each other. By Proposition 29, $\angle CAB \cong \angle ACD$ and $\angle DBA \cong \angle BDC$. Hence $\triangle ABE \cong \triangle CDE$ (by Proposition 26) which implies that $\overline{AE} \cong \overline{CE}$ and $\overline{DE} \cong \overline{BE}$. Therefore the diagonals are equal and bisect each other.

1.6.11 The argument works for the first case. This is the case in which the triangle actually is isosceles. The second case never occurs (D is never inside the triangle). The flaw lies in the third case (D is outside the triangle). If the triangle is not isosceles then either E will be outside the triangle and F will be on the edge $\overline{AC}$, or E will be on the edge $\overline{AB}$ and F will be outside. They cannot both be outside as shown in the diagram. This can be checked by drawing a careful diagram by hand or by drawing the diagram using GeoGebra (or similar software).

[2]It should be noted that the fact about rhombi can be proved using just propositions that come early in Book I and do not depend on the Fifth Postulate, whereas the proof in the next exercise requires propositions about parallelism that Euclid proves much later in Book I using his Fifth Postulate.

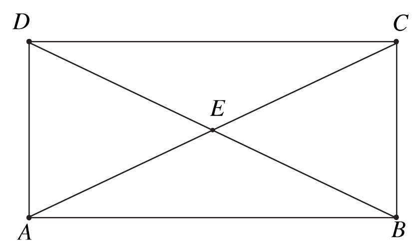

FIGURE S1.7: Exercise 1.6.10 Rectangle $\square ABCD$