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# 1 Logic and Proofs

## 1.1 Propositions and Connectives

1. (a) true (b) false (c) true (d) false  
(e) false (f) false (g) false (h) false
2. (a) Not a proposition  
(b) False proposition  
(c) Not a proposition. It would be a proposition if a value for  $x$  had been assigned.  
(d) Not a proposition. It would be a proposition if values for  $x$  and  $y$  had been assigned.  
(e) False proposition  
(f) True proposition  
(g) False proposition  
(h) True proposition  
(i) False proposition  
(j) Not a proposition. It is neither true nor false.

3. (a)

$P$	$\sim P$	$P \wedge \sim P$
T	F	T
F	T	F

(b)

$P$	$\sim P$	$P \vee \sim P$
T	F	T
F	T	T

(c)

$P$	$Q$	$\sim Q$	$P \wedge \sim Q$
T	T	F	F
F	T	F	F
T	F	T	T
F	F	T	F

(d)

$P$	$Q$	$\sim Q$	$Q \vee \sim Q$	$P \wedge (Q \vee \sim Q)$
T	T	F	T	T
F	T	F	T	F
T	F	T	T	T
F	F	T	T	F

(e)

$P$	$Q$	$\sim Q$	$P \wedge Q$	$(P \wedge Q) \vee \sim Q$
T	T	F	T	T
F	T	F	F	F
T	F	T	F	T
F	F	T	F	T

(f)

$P$	$Q$	$P \wedge Q$	$\sim (P \wedge Q)$
T	T	T	F
F	T	F	T
T	F	F	T
F	F	F	T

(g)

$P$	$Q$	$R$	$\sim Q$	$P \vee \sim Q$	$(P \vee \sim Q) \wedge R$
T	T	T	F	T	T
F	T	T	F	F	F
T	F	T	T	T	T
F	F	T	T	T	T
T	T	F	F	T	F
F	T	F	F	F	F
T	F	F	T	T	F
F	F	F	T	T	F

(h)

$P$	$Q$	$\sim P$	$\sim Q$	$\sim P \wedge \sim Q$
T	T	F	F	F
F	T	T	F	F
T	F	F	T	F
F	F	T	T	T

(i)

$P$	$Q$	$R$	$Q \vee R$	$P \wedge (Q \vee R)$
T	T	T	T	T
F	T	T	T	F
T	F	T	T	T
F	F	T	T	F
T	T	F	T	T
F	T	F	T	F
T	F	F	F	F
F	F	F	F	F

(j)

$P$	$Q$	$R$	$P \wedge Q$	$P \wedge R$	$(P \wedge Q) \vee (P \wedge R)$
T	T	T	T	T	T
F	T	T	F	F	F
T	F	T	F	T	T
F	F	T	F	F	F
T	T	F	T	F	T
F	T	F	F	F	F
T	F	F	F	F	F
F	F	F	F	F	F

4. (a) false (b) true (c) true (d) true  
 (e) false (f) false (g) false (h) false  
 (i) true (j) true (k) false (l) false

5. (a) No solution.

(b)

$P$	$Q$	$P \vee Q$	$Q \vee P$
T	T	T	T
F	T	T	T
T	F	T	T
F	F	F	F

Since the third and fourth columns are the same, the propositions are equivalent.

(c)

$P$	$Q$	$P \wedge Q$	$Q \wedge P$
T	T	T	T
F	T	F	F
T	F	F	F
F	F	F	F

Since the third and fourth columns are the same, the propositions are equivalent.

(d)

$P$	$Q$	$R$	$Q \vee R$	$P \vee (Q \vee R)$	$P \vee Q$	$(P \vee Q) \vee R$
T	T	T	T	T	T	T
F	T	T	T	T	T	T
T	F	T	T	T	T	T
F	F	T	T	T	F	T
T	T	F	T	T	T	T
F	T	F	T	T	T	T
T	F	F	F	T	T	T
F	F	F	F	F	F	F

Since the fifth and seventh columns are the same, the propositions are equivalent.

(e)

$P$	$Q$	$R$	$Q \wedge R$	$P \wedge (Q \wedge R)$	$P \wedge Q$	$(P \wedge Q) \wedge R$
T	T	T	T	T	T	T
F	T	T	T	F	F	F
T	F	T	F	F	F	F
F	F	T	F	F	F	F
T	T	F	F	F	T	F
F	T	F	F	F	F	F
T	F	F	F	F	F	F
F	F	F	F	F	F	F

Since the fifth and seventh columns are the same, the propositions are equivalent.

(f)

$P$	$Q$	$R$	$Q \vee R$	$P \wedge (Q \vee R)$	$P \wedge Q$	$P \wedge R$	$(P \wedge Q) \vee (P \wedge R)$
T	T	T	T	T	T	T	T
F	T	T	T	F	F	F	F
T	F	T	T	T	F	T	T
F	F	T	T	F	F	F	F
T	T	F	T	T	T	F	T
F	T	F	T	F	F	F	F
T	F	F	F	F	F	F	F
F	F	F	F	F	F	F	F

Since the fifth and eighth columns are the same, the propositions are equivalent.

(g)

$P$	$Q$	$R$	$Q \wedge R$	$P \vee (Q \wedge R)$	$P \vee Q$	$P \vee R$	$(P \vee Q) \wedge (P \vee R)$
T	T	T	T	T	T	T	T
F	T	T	T	T	T	T	T
T	F	T	F	T	T	T	T
F	F	T	F	F	F	T	F
T	T	F	F	T	T	T	T
F	T	F	F	F	T	F	F
T	F	F	F	T	T	T	T
F	F	F	F	F	F	F	F

Since the fifth and eighth columns are the same, the propositions are equivalent.

(h) No solution.

(i)

$P$	$Q$	$P \vee Q$	$\sim (P \vee Q)$	$\sim P$	$\sim Q$	$\sim P \wedge \sim Q$
T	T	T	F	F	F	F
F	T	T	F	T	F	F
T	F	T	F	F	T	F
F	F	F	T	T	T	T

Since the fourth and eighth columns are the same, the propositions are equivalent.

6. (a) equivalent (b) equivalent  
 (c) equivalent (d) equivalent  
 (e) equivalent (f) not equivalent  
 (g) not equivalent (h) not equivalent
7. (a)  $\sim P$ , true (b)  $P \wedge Q$ , true  
 (c)  $P \vee Q$ , true (d)  $P \vee Q \vee R$ , true
8. (a) Since  $P$  is equivalent to  $Q$ ,  $P$  has the same truth table as  $Q$ . Therefore,  $Q$  has the same truth table as  $P$ , so  $Q$  is equivalent to  $P$ .  
 (b) Since  $P$  is equivalent to  $Q$ ,  $P$  and  $Q$  have the same truth table. Since  $Q$  is equivalent to  $R$ ,  $Q$  and  $R$  have the same truth table. Thus,  $P$  and  $R$  have the same truth table so  $P$  is equivalent to  $R$ .  
 (c) Since  $P$  is equivalent to  $Q$ ,  $P$  and  $Q$  have the same truth table. That is, the truth table for  $P$  has value true on exactly the same lines that the truth table for  $Q$  has value true. Therefore the truth table for  $\sim Q$  has value false on exactly the same lines that the truth table for  $\sim P$  has the value false. Thus  $\sim Q$  and  $\sim P$  have the same truth table.
9. (a)  $(P \wedge Q) \vee (\sim P \wedge \sim Q)$  is neither.

$P$	$Q$	$\sim P$	$\sim Q$	$P \wedge Q$	$\sim P \wedge \sim Q$	$(P \wedge Q) \vee (\sim P \wedge \sim Q)$
T	T	F	F	T	F	T
F	T	T	F	F	F	F
T	F	F	T	F	F	F
F	F	T	T	F	T	T

(b)  $\sim (P \wedge \sim P)$  is a tautology.

$P$	$\sim P$	$P \wedge \sim P$	$\sim (P \wedge \sim P)$
T	F	F	T
F	T	F	T

(c)  $(P \wedge Q) \vee (\sim P \vee \sim Q)$  is a tautology.

$P$	$Q$	$\sim P$	$\sim Q$	$P \wedge Q$	$\sim P \vee \sim Q$	$(P \wedge Q) \vee (\sim P \vee \sim Q)$
T	T	F	F	T	F	T
F	T	T	F	F	T	T
T	F	F	T	F	T	T
F	F	T	T	F	T	T

(d)  $(A \wedge B) \vee (A \wedge \sim B) \vee (\sim A \wedge B) \vee (\sim A \wedge \sim B)$  is a tautology.

$A$	$B$	$\sim A$	$\sim B$	$A \wedge B$	$A \wedge \sim B$	$\sim A \wedge B$	$\sim A \wedge \sim B$	$(A \wedge B) \vee (A \wedge \sim B) \vee (\sim A \wedge B) \vee (\sim A \wedge \sim B)$
T	T	F	F	T	F	F	F	T
F	T	T	F	F	T	T	F	T
T	F	F	T	F	T	F	F	T
F	F	T	T	F	F	F	T	T

(e)  $(Q \wedge \sim P) \wedge \sim (P \wedge R)$  is neither.

$P$	$Q$	$R$	$\sim P$	$Q \wedge \sim P$	$P \wedge R$	$\sim (P \wedge R)$	$(Q \wedge \sim P) \wedge \sim (P \wedge R)$
T	T	T	F	F	T	F	F
F	T	T	T	T	F	T	T
T	F	T	F	F	T	F	F
F	F	T	T	F	F	T	F
T	T	F	F	F	F	T	F
F	T	F	T	T	F	T	T
T	F	F	F	F	F	T	F
F	F	F	T	F	F	T	F

(f)  $P \vee [(\sim Q \wedge P) \wedge (R \vee Q)]$  is neither.

$P$	$Q$	$R$	$\sim Q$	$\sim Q \wedge P$	$R \vee Q$	$[(\sim Q \wedge P) \wedge (R \vee Q)]$	$P \vee [(\sim Q \wedge P) \wedge (R \vee Q)]$
T	T	T	F	F	T	F	T
F	T	T	F	F	T	F	F
T	F	T	T	T	T	T	T
F	F	T	T	F	T	F	F
T	T	F	F	F	T	F	T
F	T	F	F	F	T	F	F
T	F	F	T	T	F	F	T
F	F	F	T	F	F	F	F

10. (a) contradiction (b) tautology  
(c) tautology (d) tautology
11. (a)  $x$  is not a positive integer.  
(b) Cleveland will lose the first game and the second game. Or, Cleveland will lose both games.  
(c)  $5 < 3$   
(d) 641,371 is not composite. Or 641,371 is prime.  
(e) Roses are not red or violets are not blue.  
(f)  $T$  is bounded and  $T$  is not compact.  
(g)  $M$  is not odd or  $M$  is not one-to-one.  
(h) The function  $f$  does not have a positive first derivative at  $x$  or does not have a positive second derivative at  $x$ .  
(i) The function  $g$  does not have a relative maximum at  $x = 2$  (deleted comma) and does not have a relative maximum at  $x = 4$ , or else  $g$  does not have a relative minimum at  $x = 3$ .  
(j)  $z < s$  or  $z \leq t$ .  
(k)  $R$  is not transitive or  $R$  is reflexive.  
(l) If the function  $g$  has a relative minimum at  $x = 2$  or  $x = 4$ , then  $g$  does not have a relative minimum at  $x = 3$ .
12. (a)  $[\sim (\sim P)] \vee [(\sim Q) \wedge (\sim S)]$   
(b)  $[Q \wedge (\sim S)] \vee \sim (P \wedge [Q \wedge (\sim S)]) \vee \sim ((\sim P \wedge Q))$ .  
(c)  $[[P \wedge (\sim Q)] \vee [(\sim P) \wedge (\sim R)]] \vee [(\sim P) \wedge S]$   
(d)  $[(\sim P) \vee ([Q \wedge (\sim P)]) \wedge Q] \vee R$ .

13. (a) i.

$A$	$B$	$A \odot B$
T	T	F
F	T	T
T	F	T
F	F	F

ii.

$A$	$B$	$A \vee B$	$A \wedge B$	$\sim (A \wedge B)$	$(A \vee B) \wedge \sim (A \wedge B)$
T	T	T	T	F	F
F	T	T	F	T	T
T	F	T	F	T	T
F	F	F	F	T	F

Since the final columns of the two tables are identical, the two propositions have the same truth table, thus they are equivalent.

(b) i.

$A$	$B$	$A \text{ NAND } B$	$A \text{ NOR } B$
T	T	F	F
F	T	T	F
T	F	T	F
F	F	T	T

ii.

$A$	$B$	$A \text{ NAND } B$	$A \text{ NOR } B$	$(A \text{ NAND } B) \vee (A \text{ NOR } B)$
T	T	F	F	F
F	T	T	F	T
T	F	T	F	T
F	F	T	T	T

Since the third and last columns are equal, the propositions are equivalent.

iii.

$A$	$B$	$A \text{ NAND } B$	$A \text{ NOR } B$	$(A \text{ NAND } B) \wedge (A \text{ NOR } B)$
T	T	F	F	F
F	T	T	F	F
T	F	T	F	F
F	F	T	T	T

Since the fourth and last columns are equal, the propositions are equivalent.

## 1.2 Conditionals and Biconditionals

1. (a) Antecedent: squares have three.  
Consequent: triangles have four sides.
- (b) Antecedent: The moon is made of cheese.  
Consequent: 8 is an irrational number.
- (c) Antecedent:  $b$  divides 3.  
Consequent:  $b$  divides 9.
- (d) Antecedent:  $f$  is differentiable.  
Consequent:  $f$  is continuous.
- (e) Antecedent:  $a$  is convergent.  
Consequent:  $a$  is bounded.
- (f) Antecedent:  $f$  is integrable.  
Consequent:  $f$  is bounded.
- (g) Antecedent:  $1 + 1 = 2$ .  
Consequent:  $1 + 2 = 3$ .



- (h) Antecedent: the fish bite.  
Consequent: the moon is full.
  - (i) Antecedent: An athlete qualifies for the Olympic team.  
Consequent: The athlete has a time of 3 minutes, 48 seconds or less ( in the event).
2. (a) Converse: If triangles have four sides, then squares have three sides.  
Contrapositive: If triangles do not have four sides, then squares do not have three sides.
- (b) Converse: If 8 is irrational, then the moon is made of cheese.  
Contrapositive: If 8 is rational, then the moon is not made of cheese.
- (c) Converse: If  $b$  divides 9, then  $b$  divides 3.  
Contrapositive: If  $b$  does not divide 9, then  $b$  does not divide 3.
- (d) Converse: If  $f$  is continuous, then  $f$  is differentiable.  
Contrapositive: If  $f$  is not continuous, then  $f$  is not differentiable.
- (e) Converse: If  $a$  is bounded, then  $a$  is convergent.  
Contrapositive: If  $a$  is not bounded, then  $a$  is not convergent.
- (f) Converse: If  $f$  is bounded, then  $f$  is integrable.  
Contrapositive: If  $f$  is not bounded, then  $f$  is not integrable.
- (g) Converse: If  $1 + 2 = 3$ , then  $1 + 1 = 2$ .  
Contrapositive: If  $1 + 1 \neq 2$ , then  $1 + 2 \neq 3$ .
- (h) Converse: If the moon is full, then fish will bite.  
Contrapositive: If the moon is not full, then fish will not bite.
- (i) Converse: A time of 3 minutes, 48 seconds or less is sufficient to qualify for the Olympic team.  
Contrapositive: If an athlete records a time that is not 3 minutes and 48 seconds or less, then that athlete does not qualify for the Olympic team.
3. (a)  $Q$  may be either true or false.  
(b)  $Q$  must be true.  
(c)  $Q$  must be false.  
(d)  $Q$  must be false.  
(e)  $Q$  must be false.
4. (a) Antecedent:  $A(x)$  is an open sentence with variable  $x$ .  
Consequent:  $\sim (\forall x)A(x)$  is equivalent to  $(\exists x) \sim A(x)$ .
- (b) Antecedent: Every even natural number greater than 2 is the sum of two primes.  
Consequent: Every odd natural number greater than 5 is the sum of three primes.
- (c) Antecedent:  $A$  is a set with  $n$  elements.  
Consequent:  $\mathcal{P}(A)$  is a set with  $2^n$  elements.
- (d) Antecedent:  $S$  is a subset of  $\mathbb{N}$  such that  $1 \in S$  and, for all  $n \in \mathbb{N}$ , if  $n \in S$ , then  $n + 1 \in S$ .  
Consequent:  $S = \mathbb{N}$ .
- (e) Antecedent:  $A$  is a finite set with  $m$  elements and  $B$  is a finite set with  $n$  elements.  
Consequent:  $\overline{\overline{A \times B}} = mn$ .

- (f) Antecedent:  $R$  is a partial order for  $A$  and  $B \subseteq A$ .  
 Consequent: If  $\sup(B)$  exists, it is unique.
- (g) Antecedent:  $A, B, C$ , and  $D$  are sets,  $f$  is a function from  $A$  to  $B$ ,  $g$  is a function from  $B$  to  $C$ , and  $h$  is a function from  $C$  to  $D$ .  
 Consequent:  $(h \circ g) \circ f = h \circ (g \circ f)$ .
- (h) Antecedent:  $A$  and  $B$  are disjoint finite sets.  
 Consequent:  $A \cup B$  is finite and  $\overline{A \cup B} = \overline{A} + \overline{B}$ .

5. (a) true (b) false (c) true (d) true  
 (e) true (f) true (g) true (h) false
6. (a) true (b) true (c) true (d) true  
 (e) false (f) true (g) false (The symbol for helium is He.)  
 (h) true (i) false (j) true (k) false

7. (a)

$P$	$Q$	$Q \wedge P$	$P \Rightarrow (Q \wedge P)$
T	T	T	T
F	T	F	T
T	F	F	F
F	F	F	T

(b)

$P$	$Q$	$\sim P$	$\sim P \Rightarrow Q$	$Q \Leftrightarrow P$	$(\sim P \Rightarrow Q) \vee (Q \Leftrightarrow P)$
T	T	F	T	T	T
F	T	T	T	F	T
T	F	F	T	F	T
F	F	T	F	T	T

(c)

$P$	$Q$	$\sim Q$	$Q \Leftrightarrow P$	$\sim Q \Rightarrow (Q \Leftrightarrow P)$
T	T	F	T	T
F	T	F	F	T
T	F	T	F	F
F	F	T	T	T

(d)

$P$	$Q$	$P \vee Q$	$P \wedge Q$	$(P \vee Q) \Rightarrow (P \wedge Q)$
T	T	T	T	T
F	T	T	F	F
T	F	T	F	F
F	F	F	F	T

(e)

$P$	$Q$	$R$	$P \wedge Q$	$Q \wedge R$	$P \vee R$	$(P \wedge Q) \vee (Q \wedge R)$	$(P \wedge Q) \vee (Q \wedge R) \Rightarrow P \vee R$
T	T	T	T	T	T	T	T
F	T	T	F	T	T	T	T
T	F	T	F	F	T	F	T
F	F	T	F	F	T	F	T
T	T	F	T	F	T	T	T
F	T	F	F	F	F	F	T
T	F	F	F	F	T	F	T
F	F	F	F	F	F	F	T

(f)

$P$	$Q$	$R$	$S$	$Q \Rightarrow S$	$Q \Rightarrow R$	$P \vee Q$	$S \vee R$	$(Q \Rightarrow S) \wedge (Q \Rightarrow R)$
T	T	T	T	T	T	T	T	T
F	T	T	T	T	T	T	T	T
T	F	T	T	T	T	T	T	T
F	F	T	T	T	T	F	T	T
T	T	F	T	T	F	T	T	F
F	T	F	T	T	F	T	T	F
T	F	F	T	T	T	T	T	T
F	F	F	T	T	T	F	T	T
T	T	T	F	F	T	T	T	F
F	T	T	F	F	T	T	T	F
T	F	T	F	T	T	T	T	T
F	F	T	F	T	T	F	T	T
T	T	F	F	F	F	T	F	F
F	T	F	F	F	F	T	F	F
T	F	F	F	T	T	T	F	T
F	F	F	F	T	T	F	F	T

$P$	$Q$	$R$	$S$	$(P \vee Q) \Rightarrow (S \vee R)$	$[(Q \Rightarrow S) \wedge (Q \Rightarrow R)] \Rightarrow [(P \vee Q) \Rightarrow (S \vee R)]$
T	T	T	T	T	T
F	T	T	T	T	T
T	F	T	T	T	T
F	F	T	T	T	T
T	T	F	T	T	T
F	T	F	T	T	T
T	F	F	T	T	T
F	F	F	T	T	T
T	T	T	F	T	T
F	T	T	F	T	T
T	F	T	F	T	T
F	F	T	F	T	T
T	T	F	F	F	T
F	T	F	F	F	T
T	F	F	F	F	F
F	F	F	F	T	T

8. (a)

$P$	$Q$	$\sim P$	$P \Rightarrow Q$	$(\sim P) \vee Q$
T	T	F	T	T
F	T	T	T	T
T	F	F	F	F
F	F	T	T	T

Since the fourth and fifth columns are the same, the propositions  $P \Rightarrow Q$  and  $(\sim P) \vee Q$  are equivalent.

(b)

$P$	$Q$	$P \Rightarrow Q$	$Q \Rightarrow P$	$P \Leftrightarrow Q$	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
T	T	T	T	T	T
F	T	T	F	F	F
T	F	F	T	F	F
F	F	T	T	T	T

Since the fifth and sixth columns are the same, the propositions  $P \Leftrightarrow Q$  and  $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$  are equivalent.

(c)

$P$	$Q$	$\sim Q$	$P \Rightarrow Q$	$\sim (P \Rightarrow Q)$	$P \wedge \sim Q$
T	T	F	T	F	F
F	T	F	T	F	F
T	F	T	F	T	T
F	F	T	T	F	F

Since the fifth and sixth columns are the same, the propositions  $\sim (P \Rightarrow Q)$  and  $P \wedge \sim Q$  are equivalent.

(d)

$P$	$Q$	$\sim P$	$\sim Q$	$P \wedge Q$	$\sim (P \wedge Q)$	$P \Rightarrow \sim Q$	$P \Rightarrow \sim Q$
T	T	F	F	T	F	F	F
F	T	T	F	F	T	T	T
T	F	F	T	F	T	T	T
F	F	T	T	F	T	T	T

Since the sixth, seventh and eighth columns are the same, all three propositions are equivalent.

(e)

$P$	$Q$	$R$	$Q \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$	$P \wedge Q$	$(P \wedge Q) \Rightarrow R$
T	T	T	T	T	T	T
F	T	T	T	T	F	T
T	F	T	T	T	F	T
F	F	T	T	T	F	T
T	T	F	F	F	T	F
F	T	F	F	T	F	T
T	F	F	T	T	F	T
F	F	F	T	T	F	T

Since the fifth and seventh columns are the same, the propositions are equivalent.

(f)

$P$	$Q$	$R$	$Q \wedge R$	$P \Rightarrow (Q \wedge R)$	$P \Rightarrow Q$	$P \Rightarrow R$	$(P \Rightarrow Q) \wedge (P \Rightarrow R)$
T	T	T	T	T	T	T	T
F	T	T	T	T	T	T	T
T	F	T	F	F	F	T	F
F	F	T	F	T	T	T	T
T	T	F	F	F	T	F	F
F	T	F	F	T	T	T	T
T	F	F	F	F	F	F	F
F	F	F	F	T	T	T	T

Since the fifth and eighth columns are the same, the propositions are equivalent.

(g)

$P$	$Q$	$R$	$P \vee Q$	$P \vee Q \Rightarrow R$	$P \Rightarrow R$	$Q \Rightarrow R$	$(P \Rightarrow R) \wedge (Q \Rightarrow R)$
T	T	T	T	T	T	T	T
F	T	T	T	T	T	T	T
T	F	T	T	T	T	T	T
F	F	T	F	T	T	T	T
T	T	F	T	F	F	F	F
F	T	F	T	F	T	F	F
T	F	F	T	F	F	T	F
F	F	F	F	T	T	T	T

Since the fifth and seventh columns are the same, the propositions are equivalent.

9. (a) yes      (b) no      (c) yes  
 (d) yes      (e) no      (f) no

10. (a)  $[(f \text{ has a relative minimum at } x_0) \wedge (f \text{ is differentiable at } x_0)] \Rightarrow (f'(x_0) = 0)$   
 (b)  $(n \text{ is prime}) \Rightarrow [(n = 2) \vee (n \text{ is odd})]$   
 (c)  $(x \text{ is irrational}) \Rightarrow [(x \text{ is real}) \wedge \sim (x \text{ is rational})]$   
 (d)  $[(x = 1) \vee (x = -1)] \Rightarrow (|x| = 1)$   
 (e)  $(x_0 \text{ is a critical point for } f) \Leftrightarrow [(f'(x_0) = 0) \vee (f'(x_0) \text{ does not exist})]$   
 (f)  $(S \text{ is compact}) \Leftrightarrow [(S \text{ is closed}) \wedge (S \text{ is bounded})]$   
 (g)  $(B \text{ is invertible}) \Leftrightarrow (\det B \neq 0)$

- (h)  $(6 \geq n - 3) \Rightarrow (n > 4) \vee (n > 10)$   
 (i)  $(x \text{ is Cauchy}) \Rightarrow (x \text{ is convergent})$   
 (j)  $(\lim_{x \rightarrow x_0} f(x) = f(x_0)) \Rightarrow (f \text{ is continuous at } x_0)$   
 (k)  $[(f \text{ is differentiable at } x_0) \wedge (f \text{ is strictly increasing at } x_0)] \Rightarrow (f'(x_0))$
11. (a) Let  $S$  be “I go to the store” and  $R$  be “It rains.” The preferred translation: is  $\sim S \Rightarrow R$  (or, equivalently,  $\sim R \Rightarrow S$ ). This could be read as “If it doesn’t rain, then I go to the store.”  
 The speaker might mean “I go to the store if and only if it doesn’t rain” ( $S \Rightarrow \sim R$ ) or possibly “If it rains, then I don’t go to the store” ( $R \Rightarrow \sim S$ ).
- (b) There are three nonequivalent ways to translate the sentence, using the symbols  $D$ : “The Dolphins make the playoffs” and  $B$ : “The Bears win all the rest of their games.” The first translation is preferred, but the speaker may have intended any of the three.  
 $\sim B \Rightarrow \sim D$  or, equivalently,  $D \Rightarrow B$   
 $\sim D \Rightarrow \sim B$  or, equivalently,  $B \Rightarrow D$   
 $\sim B \Leftrightarrow \sim D$  or, equivalently,  $B \Leftrightarrow D$
- (c) Let  $G$  be “You can go to the game” and  $H$  be “You do your homework first.”  
 It is most likely that a student and parent both interpret this statement as a biconditional,  $G \Leftrightarrow H$ .
- (d) Let  $W$  be “You win the lottery” and  $T$  be “You buy a ticket.” Of the three common interpretations for the word “unless,” only the form  $\sim T \Rightarrow \sim W$  (or, equivalently,  $W \Rightarrow T$ ) makes sense here.

12. (a)

$P$	$Q$	$R$	$P \vee Q$	$(P \vee Q) \Rightarrow R$	$\sim P \wedge \sim Q$	$\sim R \Rightarrow (\sim P \wedge \sim Q)$
T	T	T	T	T	F	T
F	T	T	T	T	F	T
T	F	T	T	T	F	T
F	F	T	F	T	T	T
T	T	F	T	F	F	F
F	T	F	T	F	F	F
T	F	F	T	F	F	F
F	F	F	F	T	T	T

Since the fifth and seventh columns are the same,  $(P \vee Q) \Rightarrow R$  and  $\sim R \Rightarrow (\sim P \wedge \sim Q)$  are equivalent.

(b)

$P$	$Q$	$R$	$P \wedge Q$	$(P \wedge Q) \Rightarrow R$	$\sim Q$	$\sim R$	$P \wedge \sim R$	$(P \wedge \sim R) \Rightarrow \sim Q$
T	T	T	T	T	F	F	F	T
F	T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	F	T
F	F	T	F	T	T	F	F	T
T	T	F	T	F	F	T	T	F
F	T	F	F	T	F	T	F	T
T	F	F	F	T	T	T	T	T
F	F	F	F	T	T	T	F	T

Since the fifth and ninth columns are the same, the propositions  $(P \wedge Q) \Rightarrow R$  and  $(P \wedge \sim R) \Rightarrow \sim Q$  are equivalent.

(c)

$P$	$Q$	$R$	$Q \wedge R$	$P \Rightarrow (Q \wedge R)$	$\sim Q \vee \sim R$	$(\sim Q \vee \sim R) \Rightarrow \sim P$
T	T	T	T	T	F	T
F	T	T	T	T	F	T
T	F	T	F	F	T	F
F	F	T	F	T	T	T
T	T	F	F	F	T	F
F	T	F	F	T	T	T
T	F	F	F	F	T	F
F	F	F	F	T	T	T

Since the fifth and seventh columns are the same, the propositions  $P \Rightarrow (Q \wedge R)$  and  $(\sim Q \vee \sim R) \Rightarrow \sim P$  are equivalent.

(d)

$P$	$Q$	$R$	$Q \vee R$	$P \Rightarrow (Q \vee R)$	$P \wedge \sim R$	$(P \wedge \sim R) \Rightarrow Q$
T	T	T	T	T	F	T
F	T	T	T	T	F	T
T	F	T	T	T	F	T
F	F	T	T	T	F	T
T	T	F	T	T	T	T
F	T	F	T	T	F	T
T	F	F	F	F	T	F
F	F	F	F	T	F	T

Since the fifth and seventh columns are the same, the propositions  $P \Rightarrow (Q \vee R)$  and  $(P \wedge \sim R) \Rightarrow Q$  are equivalent.

(e)

$P$	$Q$	$R$	$P \Rightarrow Q$	$(P \Rightarrow Q) \Rightarrow R$	$P \wedge \sim Q$	$(P \wedge \sim Q) \vee R$
T	T	T	T	T	F	T
F	T	T	T	T	F	T
T	F	T	F	T	T	T
F	F	T	T	T	F	T
T	T	F	T	F	F	F
F	T	F	T	F	F	F
T	F	F	F	T	T	T
F	F	F	T	F	F	F

Since the fifth and seventh columns are the same, the propositions  $(P \Rightarrow Q) \Rightarrow R$  and  $(P \wedge \sim Q) \vee R$  are equivalent.

(f)

$P$	$Q$	$P \Leftrightarrow Q$	$\sim P \vee Q$	$\sim Q \vee P$	$(\sim P \vee Q) \wedge (\sim Q \vee P)$
T	T	T	T	T	T
F	T	F	T	F	F
T	F	F	F	T	F
F	F	T	T	T	T

Since the third and sixth columns are the same, the propositions  $P \Leftrightarrow Q$  and  $(\sim P \vee Q) \wedge (\sim Q \vee P)$  are equivalent.

13. (a) If 6 is an even integer, then 7 is an odd integer.  
 (b) If 6 is an odd integer, then 7 is an odd integer.  
 (c) This is not possible.  
 (d) If 6 is an even integer, then 7 is an even integer. (Any true conditional statement will work here.)
14. (a) If 7 is an odd integer, then 6 is an odd integer.  
 (b) This is not possible.

- (c) This is not possible.
- (d) If 7 is an odd integer, then 6 is an odd integer. (Any false conditional statement will work here.)
15. (a) Converse: If  $f'(x_0) = 0$ , then  $f$  has a relative minimum at  $x_0$  and is differentiable at  $x_0$ . False:  $f(x) = x^3$  has first derivative 0 but no minimum at  $x_0 = 0$ .  
Contrapositive: If  $f'(x_0) \neq 0$ , then  $f$  either has no relative minimum at  $x_0$  or is not differentiable at  $x_0$ . True.
- (b) Converse: If  $n = 2$  or  $n$  is odd, then  $n$  is prime. False: 9 is odd but not prime.  
Contrapositive: If  $n$  is even and not equal to 2, then  $n$  is not prime. True.
- (c) Converse: If  $x$  is irrational, then  $x$  is real and not rational. True  
Contrapositive: If  $x$  is not irrational, then  $x$  is not real or  $x$  is rational. True
- (d) Converse: If  $|x| = 1$ , then  $x = 1$  or  $x = -1$ . True.  
Contrapositive: If  $|x| \neq 1$ , then  $x \neq 1$  and  $x \neq -1$ . True.
16. (a) tautology (b) tautology (c) contradiction  
(d) neither (e) tautology (f) neither  
(g) contradiction (h) tautology (i) contradiction  
(j) neither (k) tautology (l) neither

17. (a)

$P$	$Q$	$P \Rightarrow Q$	$\sim P$	$\sim Q$	$\sim P \Rightarrow \sim Q$
T	T	T	F	F	T
F	T	T	T	F	F
T	F	F	F	T	T
F	F	T	T	T	T

Comparison of the third and sixth columns of the truth table shows that  $P \Rightarrow Q$  and  $\sim P \Rightarrow \sim Q$  are not equivalent.

- (b) We see from the truth table in part (a) that both propositions  $P \Rightarrow Q$  and  $\sim P \Rightarrow \sim Q$  are true only when  $P$  and  $Q$  have the same truth value.
- (c) The converse of  $P \Rightarrow Q$  is  $Q \Rightarrow P$ . The contrapositive of the inverse of  $P \Rightarrow Q$  is  $\sim \sim Q \Rightarrow \sim \sim P$ , so the converse and the contrapositive of the inverse are equivalent.  
The inverse of the contrapositive of  $P \Rightarrow Q$  is also  $\sim \sim Q \Rightarrow \sim \sim P$ , so it too is equivalent to the converse.

### 1.3 Quantifiers

1. (a)  $\sim (\forall x)(x \text{ is precious} \Rightarrow x \text{ is beautiful})$  or  $(\exists x)(x \text{ is precious and } x \text{ is not beautiful})$   
(b)  $(\forall x)(x \text{ is precious} \Rightarrow x \text{ is not beautiful})$   
(c)  $(\exists x)(x \text{ is isosceles} \wedge x \text{ is a right triangle})$   
(d)  $(\forall x)(x \text{ is a right triangle} \Rightarrow x \text{ is not isosceles})$  or  $\sim (\exists x)(x \text{ is a right triangle} \wedge x \text{ is isosceles})$   
(e)  $(\forall x)(x \text{ is honest}) \vee \sim (\exists x)(x \text{ is honest})$   
(f)  $(\exists x)(x \text{ is honest}) \wedge (\exists x)(x \text{ is not honest})$   
(g)  $(\forall x)(x \neq 0 \Rightarrow (x > 0 \vee x < 0))$   
(h)  $(\forall x)(x \text{ is an integer} \Rightarrow (x > -4 \vee x < 6))$  or  $(\forall x \in \mathbf{Z})(x > -4 \vee x < 6)$

- (i)  $(\forall x)(\exists y)(x > y)$
  - (j)  $(\forall x)(\exists y)(x < y)$
  - (k)  $(\forall x)(\forall y)[(x \text{ is an integer} \wedge y \text{ is an integer} \wedge y > x) \Rightarrow (\exists z)(y > z > x)]$  or  $(\forall x \in \mathbb{Z})(\forall y \in \mathbb{Z})[y > x \Rightarrow (\exists z)(y > z > x)]$
  - (l)  $(\exists x)(x \text{ is a positive integer and } x \text{ is smaller than all other positive integers})$  or  $(\exists x)(x \text{ is a positive integer and } (\forall y)(y \text{ is a positive integer} \Rightarrow x \leq y))$  or  $(\exists x \in \mathbb{Z})[x > 0 \wedge (\forall y \in \mathbb{Z})(y > 0 \Rightarrow y > x)]$
  - (m)  $(\forall x)(\sim (\forall y)(x \text{ loves } y))$  or  $\sim (\exists x)(\forall y)(x \text{ loves } y)$  or  $(\forall x)(\exists y)(x \text{ does not love } y)$
  - (n)  $(\forall x)(\exists y)(x \text{ loves } y)$
  - (o)  $(\forall x)(x > 0 \Rightarrow (\exists! y)(2^y = x))$
2. (a)  $(\forall x)(x \text{ is precious} \Rightarrow x \text{ is beautiful})$   
All precious stones are beautiful.
- (b)  $(\exists x)(x \text{ is precious} \wedge x \text{ is beautiful})$   
There is a beautiful precious stone, or Some precious stones are beautiful.
- (c)  $\sim (\exists x)(x \text{ is isosceles and } x \text{ is a right triangle})$  or  $(\forall x)(x \text{ is not isosceles or } x \text{ is not a right triangle})$  or  $(\forall x)(x \text{ is right triangle} \Rightarrow x \text{ is not isosceles})$  or  $(\forall x)(x \text{ is isosceles} \Rightarrow x \text{ is not a right triangle})$ .  
There is no isosceles right triangle.
- (d)  $(\exists x)(x \text{ is isosceles} \wedge x \text{ is a right triangle})$   
There is an isosceles right triangle.
- (e)  $(\exists x)(x \text{ is dishonest}) \wedge (\exists x)(x \text{ is dishonest})$   
Some people are honest and some people are dishonest.
- (f)  $(\forall x)(x \text{ is honest}) \vee (\forall x)(x \text{ is dishonest})$   
All people are honest or no one is honest.
- (g)  $(\exists x)(x \neq 0 \wedge x \text{ is not positive} \wedge x \text{ is not negative})$   
There is a nonzero real number that is neither positive nor negative.
- (h)  $(\exists x)(x \text{ is an integer} \wedge x \leq -4 \wedge x \geq 6)$  or  $(\exists x \in \mathbb{Z})(x \leq -4 \wedge x \geq 6)$   
There is an integer that is less than or equal to  $-4$  and greater than or equal to  $6$ .
- (i)  $(\exists x)(\forall y)(x \leq y)$   
Some integer is less than or equal to every integer, or There is a smallest integer.
- (j)  $(\exists x)(\forall y)(x \geq y)$   
Some integer is greater than every other integer, or There is a largest integer.
- (k)  $(\exists x)(\exists y)[x \text{ is an integer} \wedge y \text{ is an integer } y > x \wedge (\forall z)(z \leq y \vee x \leq z)]$  or  $(\exists x \in \mathbb{Z})(\exists y \in \mathbb{Z})[y > x \wedge (\forall z)(z \leq y \vee x \leq z)]$   
There is an integer  $x$  and a larger integer  $y$  such that there is no real number between them.
- (l)  $(\forall x)(x \text{ is a positive integer} \Rightarrow (\exists y)(y \text{ is a positive integer} \wedge x > y))$  or  $(\forall x \in \mathbb{Z})[x \leq 0 \vee (\exists y \in \mathbb{Z})(y > 0 \wedge x > y)]$ . For every positive integer there is a smaller positive integer.  
Or,  $\sim (\exists x)(x \text{ is a positive integer} \wedge (\forall y)(y \text{ is a positive integer} \Rightarrow x \leq y))$  or  $\sim (\exists x \in \mathbb{Z})[x > 0 \wedge (\forall y \in \mathbb{Z})(y > 0 \Rightarrow y > x)]$   
There is no smallest positive integer.



- (m)  $(\exists x)(\forall y)(x \text{ loves } y)$   
There is someone who loves everyone.
- (n)  $(\exists x)(\forall y)(x \text{ does not loves } y) \text{ or } \sim(\forall x)(\exists y)(x \text{ loves } y)$ .  
Somebody doesn't love anyone.
- (o)  $(\exists x)(x > 0 \wedge \sim(\exists y)(2^y = x) \vee (\exists y)(\exists z)[y \neq z \wedge 2^y = x \wedge 2^z = x])$   
There is a positive real number  $x$  for which there is no unique real number  $y$  such that  $2^y = x$ .  
There is a nonzero complex number such that either every product of that number with any complex number is different from  $\pi$ , or there are at least two different complex numbers whose products with the given number are equal to  $\pi$ .
3. (a)  $(\exists k)(k \text{ is an integer } \wedge x = 2k) \text{ or } (\exists k \in \mathbb{Z})(x = 2k)$   
(b)  $(\exists j)(j \text{ is an integer } \wedge x = 2j + 1) \text{ or } (\exists j \in \mathbb{Z})(x = 2j + 1)$   
(c)  $(\exists k)(k \text{ is an integer } \wedge b = ak) \text{ or } (\exists k \in \mathbb{Z})(b = ak)$   
(d)  $n \neq 1 \wedge (\forall m \in \mathbb{Z})(m \text{ divides } n \Rightarrow (m = 1 \vee m = n))$   
(e)  $n \neq 1 \wedge (\exists m \in \mathbb{Z})(m \text{ divides } n \wedge (m \neq 1 \vee m \neq n))$
4. (a)  $(\forall x, y \in A)(xRy \Rightarrow yRx)$   
(b)  $(\forall x, y, z \in A)(xRy \wedge yRz \Rightarrow xRz)$   
(c)  $(\forall x, y \in A)(f(x) = f(y) \Rightarrow x = y)$   
(d)  $(\forall x, y \in A)(x \cdot y = y \cdot x)$
5. The first interpretation may be translated as  
 $(\forall x)[x \text{ is a person} \Rightarrow (\forall y)(y \text{ is a tax} \Rightarrow x \text{ dislikes } y)]$ .  
The other sentences may be translated as  
 $(\forall x)[x \text{ is a person} \Rightarrow (\exists y)(y \text{ is a tax} \Rightarrow x \text{ dislikes } y)]$ .  
 $(\exists x)[x \text{ is a person} \Rightarrow (\forall y)(y \text{ is a tax} \Rightarrow x \text{ dislikes } y)]$ .  
 $(\exists x)[x \text{ is a person} \Rightarrow (\exists y)(y \text{ is a tax} \Rightarrow x \text{ dislikes } y)]$ .
6. (a)  $T, U, V$  and  $W$       (b)  $T$       (c)  $T, U, V$       (d)  $T$
7. (a) **Proof.** Let  $U$  be any universe. The sentences  $\sim(\exists x)A(x)$  is true in  $U$   
iff  $(\exists x)A(x)$  is false in  $U$   
iff the truth set for  $A(x)$  is empty  
iff the truth set for  $\sim A(x)$  is  $U$  iff  $(\forall x) \sim A(x)$  is true in  $U$ .
- (b) Let  $A(x)$  be an open sentence with variable  $x$ . Then  $\sim A(x)$  is an open sentence with variable  $x$ , so we may apply part (a) of Theorem 1.3.1(b). Thus  $\sim(\forall x) \sim A(x)$  is equivalent to  $(\exists x) \sim\sim A(x)$ , which is equivalent to  $(\exists x)A(x)$ . Therefore  $\sim(\exists x)A(x)$  is equivalent to  $\sim\sim(\forall x) \sim A(x)$ , which is equivalent to  $(\forall x) \sim A(x)$ .
8. (a) false      (b) true      (c) false      (d) true  
(e) false      (f) true      (g) false      (h) true  
(i) true      (j) false      (k) false      (l) true
9. (a) Every natural number is greater than or equal to 1.  
(b) Exactly one real number is both nonnegative and nonpositive.  
(c) Every natural number that is prime and different from 2 is odd.  
(d) There is exactly one real number whose natural logarithm is 1.

- (e) There is no real number whose square is negative.  
 (f) There exists a unique real number whose square is 0.  
 (g) For every natural number, if the number is odd, then its square is odd.
10. (a) true (b) false (c) false (d) false  
 (e) true (f) false (g) true (h) false  
 (i) false (j) true (k) false
11. (a) Let  $U$  be any universe and  $A(x)$  be an open sentence. Suppose  $(\exists!x)A(x)$  is true in  $U$ . Then the truth set for  $A(x)$  has exactly one element, so the truth set for  $A(x)$  is nonempty. Thus  $(\exists x)A(x)$  is true in  $U$ .  
 (b) Let  $A(x)$  be the sentence  $x^2 = 1$  and let the universe be the real numbers. Then the truth set for  $A(x)$  is  $\{1, -1\}$  so  $(\exists x)A(x)$  is true but  $(\exists!x)A(x)$  is false in  $U$ .  
 (c) Let  $U$  be any universe and suppose  $(\exists!x)A(x)$  is true. Then the truth set for  $A(x)$  contains exactly one element  $x_0$ . As in part (a),  $(\exists x)A(x)$  is true. Suppose  $u$  and  $z$  are in  $U$  and  $A(y)$  and  $A(z)$  are true. Then  $u$  and  $z$  must both be  $x_0$ , so  $y = z$ . Thus  $(\exists x)A(x) \wedge (\forall y)(\forall z)(A(y) \wedge A(z) \Rightarrow y = z)$  is true.  
 On the other hand, suppose  $(\exists x)A(x) \wedge (\forall y)(\forall z)(A(y) \wedge A(z) \Rightarrow y = z)$  is true in  $U$ . Since  $(\exists x)A(x)$  is true, the truth set for  $A(x)$  contains at least one element. Since  $(\forall y)(\forall z)(A(y) \wedge A(z) \Rightarrow y = z)$  is true, the truth set for  $A(x)$  contains only one element. Thus  $(\exists!x)A(x)$  is true in  $U$ .  
 (d) Let  $U$  be any universe. Suppose  $(\exists!x)A(x)$  is true in  $U$ . Then the truth set for  $A(x)$  contains exactly one element,  $x_0$ . Then for every  $y$  in  $U$ , if  $A(y)$  then  $x_0 = y$ . Thus  $x_0$  is in the truth set of  $A(x) \wedge (\forall y)(A(y) \Rightarrow x = y)$ , so  $(\exists x)[A(x) \wedge (\forall y)(A(y) \Rightarrow x = y)]$  is true in  $U$ . Conversely, suppose  $(\exists x)[A(x) \wedge (\forall y)(A(y) \Rightarrow x = y)]$  is true in  $U$ . Let  $x_0$  be an element in the truth set of  $A(x) \wedge (\forall y)(A(y) \Rightarrow x = y)$ . Then  $x_0$  is the only element in the truth set of  $A(x)$ . Thus  $(\exists!x)A(x)$  is true in  $U$ .  
 (e)  $(\forall x)(\sim A(x) \vee (\exists y)(\exists z)(A(y) \wedge A(z) \wedge y \neq z))$
12. (a)  $f$  is continuous at  $a$  iff  $(\forall \epsilon)[\epsilon > 0 \Rightarrow (\exists \delta)(\delta > 0 \wedge (\forall x)(|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon)]$   
 (b) Let the universe be the set  $\mathbb{R}$  of real numbers, and let  $f$  be a function from  $\mathbb{R}$  to  $\mathbb{R}$ . The Mean Value Theorem asserts that  $(\forall a)(\forall b)((a < b) \wedge (f \text{ is continuous on } [a, b] \wedge (f \text{ is differentiable on } (a, b))) \Rightarrow$

$$(\exists c)[(a < c < b) \wedge (f'(c) = \frac{f(b) - f(a)}{b - a})],$$

where “ $f$  is continuous on  $[a, b]$ ” means:

$(\forall x_0)(a \leq x_0 \leq b \Rightarrow (\forall \epsilon)[\epsilon > 0 \Rightarrow (\forall \delta)(\delta > 0 \wedge (\forall x)(|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon)])$  and “ $f$  is differentiable on  $(a, b)$ ” means:

$(\forall x_0)(a < x_0 < b \Rightarrow (\exists d)[f'(x_0) = d])$ .

- (c) Let the universe be the set  $\mathbb{R}$  of real numbers, and let  $f$  be a function from  $\mathbb{R}$  to  $\mathbb{R}$ . Then  $\lim_{x \rightarrow a} f(x) = L$  means:  $(\forall \epsilon)(\epsilon > 0 \Rightarrow (\exists \delta)[\delta > 0 \wedge (\forall x)(|x - a| < \delta \Rightarrow |f(x) - L| < \epsilon)])$   
 (d) A denial of “ $f$  is continuous at  $a$ ” is:  $(\exists \epsilon > 0)(\forall \delta)(\delta > 0 \Rightarrow (\exists x)(|x - a| < \delta \wedge |f(x) - f(a)| \geq \epsilon))$ .  
 A denial of the Mean Value Theorem is:  $(\exists a)(\exists b)[a < b \wedge f \text{ is continuous in } [a, b] \wedge f \text{ is differentiable on } (a, b) \wedge (\forall c)(a < c < b \Rightarrow f'(c) \neq \frac{f(b) - f(a)}{b - a})]$   
 A denial of “ $\lim_{x \rightarrow a} f(x) = L$ ” is:  $(\exists \epsilon)(\epsilon > 0 \wedge (\forall \delta)[\delta > 0 \Rightarrow (\exists x)(|x - a| < \delta \wedge |f(x) - L| \geq \epsilon)])$

13. (a) This is not a denial. If the universe has only one element  $a$  and  $P(a)$  is true, then both the statement and  $(\exists!x)P(x)$  are true.  
 (b) This is a denial.  
 (c) This is a denial.  
 (d) This is not a denial. If the universe has only one element  $a$  and  $P(a)$  is false, then both the statement and  $(\exists!x)P(x)$  are false.  
 (e) This statement is not a denial. If the universe has more than one element the statement implies the negation of  $(\exists!x)P(x)$ , but if  $(\forall x)P(x)$ , then both the statement and  $(\exists!x)P(x)$  are false.
14. For every backwards E, there exists an upside down A! [This is a joke.]

## 1.4 Basic Proof Methods I

1. (a) Suppose  $(G, *)$  is a cyclic group.  
 $\vdots$   
 Thus,  $(G, *)$  is abelian.  
 Therefore, if  $(G, *)$  is a cyclic group, then  $(G, *)$  is abelian.
- (b) Suppose  $B$  is a nonsingular matrix.  
 $\vdots$   
 Thus, the determinant of  $B$  is not zero.  
 Therefore, if  $B$  is a nonsingular matrix, then the determinant of  $B$  is not zero.
- (c) Suppose  $A$  is a subset of  $B$  and  $B$  is a subset of  $C$ .  
 $\vdots$   
 Thus,  $A$  is a subset of  $C$ .  
 Therefore, if  $A$  is a subset of  $B$  and  $B$  is a subset of  $C$ , then  $A$  is a subset of  $C$ .
- (d) Suppose the maximum value of the differentiable function  $f$  on the closed interval  $[a, b]$  occurs at  $x_0$ .  
 $\vdots$   
 Thus, either  $x_0 = a$  or  $x_0 = b$  or  $f'(x_0) = 0$ . Therefore, if the maximum value of the differentiable function  $f(x)$  on the closed interval  $[a, b]$  occurs at  $x_0$ , then either  $x_0 = a$  or  $x_0 = b$  or  $f'(x_0) = 0$ .
- (e) Let  $A$  be a diagonal matrix. Suppose all the diagonal entries of  $A$  are nonzero.  
 $\vdots$   
 Then  $A$  is invertible. Therefore  $A$  is invertible whenever all its nonzero entries are nonzero.
2. If  $A$  and  $B$  are invertible matrices, then  $AB$  is invertible.
  - (a) Suppose that  $A$  and  $B$  are invertible matrices.  
 $\vdots$   
 Thus,  $AB$  is invertible.  
 Therefore, if  $A$  and  $B$  are invertible matrices, then the product  $AB$  is invertible.

(b) Suppose  $AB$  is invertible.

$\vdots$

Thus,  $A$  and  $B$  are both invertible.

Therefore, if  $AB$  is invertible, then  $A$  and  $B$  are both invertible.

3. One could construct a truth table with 16 rows and observe that every row has the value true for the main connective  $\Rightarrow$ . However, it is also correct to show, without actually making the truth table, that no row could have the value F for the main connective  $\Rightarrow$ . Suppose this connective had the value F. Then the antecedent  $[(\sim B \Rightarrow M) \wedge \sim L \wedge (\sim M \vee L)]$  must have the value T and the consequent  $B$  must have the value F. Then each of  $\sim B \Rightarrow M$ ,  $\sim L$  and  $\sim M \vee L$  has value T. Then  $L$  must have the value F, and since  $\sim M \vee L$  has value T,  $M$  has the value F. Since  $\sim B$  has the value T and  $M$  has the value F,  $\sim B \Rightarrow M$  has the value F. But  $\sim B \Rightarrow M$  has value T. This contradiction shows that every row of the truth table has value T, so the propositional form is a tautology.
4. (a) Professor Plum. The crime took place in the library, not the kitchen. By fact (i), if the crime did not take place in the kitchen, then Professor Plum is guilty. Therefore Professor Plum is guilty.
- (b) Miss Scarlet. The crime did not take place in the library. By fact (iv), the weapon was the candlestick. By fact (iii) Miss Scarlet is not innocent.
- (c) Professor Plum. The crime was committed at noon with the revolver. By (iii) Miss Scarlet is innocent. By fact (v) either Miss Scarlet or Professor Plum is guilty. Therefore Professor Plum is guilty.
- (d) Miss Scarlet and Professor Plum. The crime took place at midnight in the conservatory. By fact (ii) Professor Plum is guilty. The crime did not take place in the library. By fact (iv), the weapon was the candlestick. By fact (iii) Miss Scarlet is guilty also.
5. (a) Suppose that  $x$  and  $y$  are even. Then there are integers  $n$  and  $m$  such that  $x = 2n$  and  $y = 2m$ . By substitution,  $x + y = 2n + 2m = 2(n + m)$ . Since  $x + y$  is the product of 2 and an integer,  $x + y$  is even.
- (b) Suppose that  $x$  is an even integer, and  $y$  is an integer. Then there is an integer  $k$  such that  $x = 2k$ . Then  $xy = (2k)y = 2(ky)$ . Thus  $xy$  is twice the integer  $ky$ , so  $xy$  is even.
- (c) Suppose that  $x$  and  $y$  are even integers. Then there exist integers  $n$  and  $m$  such that  $x = 2n$  and  $y = 2m$ . Therefore,  $xy = 2n \cdot 2m = 4nm$ . Since  $nm$  is an integer,  $xy$  is divisible by 4.
- (d) Suppose that  $x$  and  $y$  are even integers. Then there are integers  $k$  and  $m$  such that  $x = 2k$  and  $y = 2m$ . Then  $3x - 5y = 3(2k) - 5(2m) = 2(3k - 5m)$ . Since  $3k - 5m$  is an integer and  $3x - 5y = 2(3k - 5m)$ ,  $3x - 5y$  is even.
- (e) Suppose that  $x$  and  $y$  are odd integers. Then there exist integers  $n$  and  $m$  such that  $x = 2n + 1$  and  $y = 2m + 1$ . By substitution,  $x + y = (2n + 1) + (2m + 1) = 2(n + m + 1)$ . Since  $x + y$  is twice an integer,  $x + y$  is even.
- (f) Then there exist integers  $k$  and  $m$  such that  $x = 2k + 1$  and  $y = 2m + 1$ . Then  $3x - 5y = 3(2k + 1) - 5(2m + 1) = 2(3k - 5m - 1)$ . Since  $3x - 5y = 2(3k - 5m - 1)$  and  $3k - 5m - 1$  is an integer, we conclude that  $3x - 5y$  is even.
- (g) Then there are integers  $k$  and  $m$  such that  $x = 2k + 1$  and  $y = 2m + 1$ . Then  $xy = (2k + 1)(2m + 1) = 2(km + k + m) + 1$ . Since  $km + k + m$  is an integer,  $xy$  is odd.

- (h) Suppose that  $x$  is even and  $y$  is odd. Then there exist integers  $n$  and  $m$  such that  $x = 2n$  and  $y = 2m + 1$ . Therefore,  $x + y = (2n) + (2m + 1) = 2(n + m) + 1$ . Since  $n + m$  is an integer,  $x + y$  is odd.
- (i) Suppose that  $x$  is even and  $y$  is odd. Then there exist integers  $k$  and  $m$  such that  $x = 2k$  and  $y = 2m + 1$ . Then  $xy = (2k)(2m + 1) = 4km + 2k = 2(2km + 1)$ . Since  $k + m$  is an integer and  $x - y = 2(k + m) + 1$ , we conclude that  $xy$  is odd.
6. (a) If  $a = 0$  or  $b = 0$ , then  $|ab| = 0 = |a||b|$ .  
Otherwise there are four cases.  
Case 1. If  $a > 0$  and  $b > 0$ , then  $|a| = a$  and  $|b| = b$ . Also,  $ab > 0$ , so  $|ab| = ab = |a||b|$ .  
Case 2. If  $a > 0$  and  $b < 0$ , then  $|a| = a$  and  $|b| = -b$ . Also,  $ab < 0$ , so  $|ab| = -ab = a(-b) = |a||b|$ .  
Case 3. If  $a < 0$  and  $b > 0$ , then  $|a| = -a$  and  $|b| = b$ . Also  $ab < 0$ , so  $|ab| = -ab = (-a)b = |a||b|$ .  
Case 4. If  $a < 0$  and  $b < 0$ , then  $|a| = -a$  and  $|b| = -b$ . Also  $ab > 0$ , so  $|ab| = ab = (-a)(-b) = |a||b|$ .  
In every case,  $|ab| = |a||b|$ .
- (b) Case 1. Let  $a - b = 0$ . Then  $b - a = 0$ , so  $|a - b| = 0 = |b - a|$ .  
Case 2. Let  $a - b > 0$ . Then  $b - a < 0$ , so  $|a - b| = a - b = -(b - a) = |b - a|$ .  
Case 3. Let  $a - b < 0$ . Then  $b - a > 0$ , so  $|a - b| = -(a - b) = b - a = |b - a|$ .  
Thus  $|a - b| = |b - a|$  in every case.
- (c) If  $a = 0$ , then  $|\frac{a}{b}| = 0 = \frac{|a|}{|b|}$ . Otherwise there are four cases.  
Case 1. Let  $a > 0$  and  $b > 0$ . Then  $|\frac{a}{b}| = \frac{a}{b} = \frac{|a|}{|b|}$ .  
Case 2. Let  $a > 0 > b$ . Then  $\frac{a}{b} < 0$ , so  $|\frac{a}{b}| = -\frac{a}{b} = \frac{a}{(-b)} = \frac{|a|}{|b|}$ .  
Case 3. Let  $a < 0 < b$ . Then  $\frac{a}{b} < 0$ , so  $|\frac{a}{b}| = -\frac{a}{b} = \frac{(-a)}{b} = \frac{|a|}{|b|}$ .  
Case 4. Let  $a < 0$  and  $b < 0$ . Then  $\frac{a}{b} > 0$ , so  $|\frac{a}{b}| = \frac{a}{b} = \frac{(-a)}{(-b)} = \frac{|a|}{|b|}$ .
- (d) Case 1:  $a \geq 0$ . There are three subcases.  
Subcase 1a:  $b \geq 0$ . Then  $a + b \geq 0$ , so  $|a + b| = a + b = |a| + |b|$ .  
Subcase 1b:  $b < 0$  and  $a \geq -b$ . Then  $a + b \geq 0$ , so  $|a + b| = a + b < a + (-b) = |a| + |b|$ .  
Subcase 1c:  $b < 0$  and  $a < -b$ . Then  $a + b < 0$ , so  $|a + b| = -(a + b) = -a - b \leq a - b = |a| + |b|$ .  
Case 2:  $a < 0$ . There are three subcases.  
Subcase 2a:  $b < 0$ . Then  $a + b < 0$ , so  $|a + b| = -(a + b) = (-a) + (-b) = |a| + |b|$ .  
Subcase 2b:  $-a > b \geq 0$ . Then  $a + b < 0$ , so  $|a + b| = -(a + b) = (-a) + (-b) \leq -a + b = |a| + |b|$ .  
Subcase 2c:  $b \geq -a > 0$ . Then  $a + b \geq 0$ , so  $|a + b| = a + b < -a + b = |a| + |b|$ .  
In every case,  $|a + b| \leq |a| + |b|$ .
- (e) Assume  $|a| \leq b$ . Then there are two cases to consider.  
Case 1:  $a \geq 0$ . Since  $b \geq |a| \geq 0$ , we have  $-b \leq 0 \leq a = |a| \leq b$ , so  $-b \leq a \leq b$ .  
Case 2:  $a < 0$ . Then  $-a = |a| \leq b$ , so  $-b \leq a < 0 < -a \leq b$  and thus  $-b \leq a \leq b$ .  
Therefore  $|a| \leq b$  implies  $-b \leq a \leq b$ .

- (f) Assume  $-b \leq a \leq b$ . There are two cases.  
 Case 1:  $a \geq 0$ . Then  $a = |a|$ , so  $|a| \leq b$ .  
 Case 2:  $a < 0$ . Then  $-a = |a|$ , so  $|a| = -a \leq -(-b) = b$ .  
 Thus  $-b \leq a \leq b$  implies  $|a| \leq b$ .
7. (a) Suppose  $a$  is an integer. Then  $2a - 1 = 2a - 2 + 1 = 2(a - 1) + 1$ . Since  $a - 1$  is an integer,  $2a - 1$  is odd.
- (b) Let  $a$  be an integer. Suppose  $a$  is even. Then  $a = 2k$  for some integer  $k$ . Therefore  $a + 1 = 2k + 1$ , so  $a + 1$  is odd.
- (c) Assume that  $a$  is an odd integer. Then for some integer  $k$ ,  $a = 2k + 1$ . Then  $a + 2 = 2k + 3 = 2(a + 1) + 1$ . Since  $a + 1$  is an integer,  $a + 2$  is odd.
- (d) Let  $a$  be an integer. If  $a$  is even, then by Exercise 7(b)  $a + 1$  is odd. By Exercise 5(i)  $a(a + 1)$  is even. On the other hand, if  $a$  is odd, then by Exercise 5(e)  $a + 1$  is even. Then, again by Exercise 5(i),  $a(a + 1)$  is even.
- (e) Let  $a$  be an integer. Then  $a = 1 \cdot a$ , so 1 divides  $a$ .
- (f) Let  $a$  be an integer. Then  $a = a \cdot 1$ , so  $a$  divides  $a$ .
- (g) Suppose  $a$  and  $b$  are positive integers and  $a$  divides  $b$ . Then for some integer  $k$ ,  $b = ka$ . Since  $b$  and  $a$  are positive,  $k$  must also be positive. Since  $k$  is also an integer,  $1 \leq k$ . Therefore,  $a = a \cdot 1 \leq a \cdot k = b$ , so  $a \leq b$ .
- (h) Let  $a$  and  $b$  be integers. Suppose that  $a$  divides  $b$ . Then  $b = ka$  for some integer  $k$ , so  $bc = kac = (kc)a$ . Since  $kc$  is an integer,  $a$  divides  $bc$ .
- (i) Suppose  $a$  and  $b$  are positive integers and  $ab = 1$ . Then  $a$  divides 1 and  $b$  divides 1. By part (g),  $a \leq 1$  and  $b \leq 1$ . But  $a$  and  $b$  are positive integers, so  $a = 1$  and  $b = 1$ .
- (j) Let  $a$  and  $b$  be positive integers. Suppose  $a$  divides  $b$  and  $b$  divides  $a$ . Then there is a positive integer  $n$  such that  $an = b$  and a positive integer  $m$  such that  $bm = a$ . Thus  $a = bm = (an)m = a(nm)$ . Then  $nm = 1$ , so  $n = 1$  and  $m = 1$  by part (i). Since  $n = 1$  and  $an = b$ ,  $a = b$ .
- (k) Let  $a$ ,  $b$ , and  $c$  be integers. Suppose  $a$  divides  $b$  and  $c$  divides  $d$ . Then  $b = ka$  and  $d = jc$  for some integers  $k$  and  $j$ . Thus  $bd = (ka)(jc) = (kj)(ac)$ , and  $kj$  is an integer, so  $ac$  divides  $bd$ .
- (l) Let  $a$ ,  $b$ , and  $c$  be integers. Suppose  $ab$  divides  $c$ . Then  $c = k(ab)$  for some integer  $k$ . Thus  $c = (kb)a$ , and  $kb$  is an integer, so  $a$  divides  $c$ .
- (m) Let  $a$ ,  $b$ , and  $c$  be integers. Suppose  $ac$  divides  $bc$ . Then there is an integer  $k$  such that  $(ac)k = bc$ . Thus  $kac = bc$ , so that  $ka = b$ . Therefore  $a$  divides  $b$ .
8. (a) Case 1:  $n$  is even. Then  $n = 2k$  for some natural number  $k$ , so
- $$n^2 + n + 3 = (2k)^2 + (2k) + 3 = 4k^2 + 2k + 2 + 1 = 2(2k^2 + k + 1) + 1.$$
- Since  $2k^2 + k + 1$  is an integer,  $n^2 + n + 3$  is odd.
- Case 2:  $n$  is odd. Then  $n = 2k + 1$  for some natural number  $k$ , so
- $$\begin{aligned} n^2 + n + 3 &= (2k + 1)^2 + (2k + 1) + 3 = 4k^2 + 4k + 1 + 2k + 1 + 3 \\ &= 4k^2 + 6k + 5 = 2(2k^2 + 3k) + 1. \end{aligned}$$
- (b) By Exercise 7(d),  $n^2 + n = n(n + 1)$  is even. Since  $n^2 + n$  is even and 3 is odd, by Exercise 5(h),  $n^2 + n + 3$  is odd.

9. (a) We want to show  $\frac{x+y}{2} \geq \sqrt{xy}$ , which could be derived from  $(\frac{x+y}{2})^2 \geq xy$ , which would follow from  $(x+y)^2 \geq 4xy$ , which would follow from  $x^2 + 2xy + y^2 \geq 4xy$ , which would follow from  $x^2 - 2xy + y^2 \geq 0$ , which would follow from  $(x-y)^2 \geq 0$ .

**Proof.** Suppose  $x$  and  $y$  are nonnegative real numbers. Then  $(x-y)^2 \geq 0$ , so  $x^2 - 2xy + y^2 \geq 0$ . Thus  $x^2 + 2xy + y^2 \geq 4xy$ , so  $(\frac{x+y}{2})^2 \geq xy$ . Since  $x$  and  $y$  are nonnegative real numbers,  $\sqrt{(x+y)^2} = x+y$  and  $\sqrt{xy}$  is a real number. Therefore  $\frac{x+y}{2} \geq \sqrt{xy}$ . We used the fact that  $x$  and  $y$  are nonnegative in the penultimate sentence in the proof.

- (b) We want to show  $a$  divides  $3c$ , which would follow if  $a$  divides  $c$ . To show  $a$  divides  $c$ , we could write  $c$  as the difference or sum of two quantities divisible by  $c$ .

**Proof.** Suppose  $a$  divides  $b$  and  $a$  divides  $b+c$ . Then using the theorem (proved as an example) on page 34,  $a$  divides the difference  $(b+c) - b = c$ . Then  $a$  divides their difference  $(b+c) - b = c$ .

- (c) We want to show  $ax^2 + bx + c = 0$  has two real solutions. This would follow if the discriminant  $b^2 - 4ac > 0$ .

**Proof.** Suppose  $ab > 0$  and  $bc < 0$ . Then the product  $ab^2c < 0$ . Since  $b^2 \geq 0$ ,  $ac < 0$ . Then  $-4ac > 0$ , so  $b^2 - 4ac > 0$ . Therefore by the discriminant test, the equation  $ax^2 + bx + c = 0$  has two real solutions.

- (d) We want to show  $2x + 5 < 11$ , which would follow from  $2x < 6$  or  $x < 3$ .

**Proof.** Suppose  $x^3 + 2x^2 < 0$ . Then  $x^2(x+2) < 0$ , so  $x+2 < 0$ . Thus  $x < -2$ , so  $x < 3$ . Therefore  $2x < 6$ , so  $2x + 5 < 11$ .

- (e) To show that the triangle is a right triangle, we want to show  $c^2 = a^2 + b^2$ .

**Proof.** Suppose a triangle has sides of length  $a$ ,  $b$ , and  $c$ , where  $c = \sqrt{2ab}$  and  $a = b$ . Then  $c^2 = (\sqrt{2ab})^2 = 2ab = 2a^2 = a^2 + a^2 = a^2 + b^2$ . Therefore the triangle is a right triangle.

10. (a) Suppose  $A > C > B > 0$ . Multiplying by the positive numbers  $C$  and  $B$ , we have  $AC > C^2 > BC$  and  $BC > B^2$ , so  $AC > B^2$ .  $AC$  is positive, so  $4AC > AC$ . Therefore  $4AC > B^2$ , so  $B^2 - 4AC < 0$ . Thus the graph must be an ellipse.
- (b) Assume  $AC < 0$ . Then  $-4AC > 0$ , so  $B^2 - 4AC > 0$ . Thus the graph is a hyperbola.
- Now assume  $B < C < 4A < 0$ . Then  $-B > -C > -4A > 0$ , so  $B^2 = (-B)(-B) > (-B)(-C) = BC$  and  $BC > 4AC$ . Thus  $B^2 - 4AC > 0$ , so the graph is a hyperbola.
- (c) Assume that the graph is a parabola. Then  $B^2 - 4AC = 0$ , so  $B^2 = 4AC$ . Assume further that  $BC \neq 0$ . Then  $C \neq 0$ , so  $A = \frac{B^2}{4C}$ .
11. (a) F. This proof, while it appears to have the essence of the correct reasoning, has too many gaps. The first “sentence” is incomplete, and the steps are not justified. The steps could be justified either by using the definitions or by referring to previous examples and exercises.
- (b) C. If  $a$  divides both  $b$  and  $c$ , then there are integers  $q_1$  and  $q_2$  such that  $b = aq_1$  and  $c = aq_2$ , but  $q_1$  and  $q_2$  are not necessarily the same number!
- (c) C. It looks as if the author of this “proof” assumed that  $x + \frac{1}{x} \geq 2$ . The proof could be fixed by beginning with the (true) statement that  $(x-1)^2 \geq 0$  and ending with the conclusion that  $x + \frac{1}{x} \geq 2$ .

- (d) F. This is a proof that if  $m$  is odd, then  $m^2$  is odd. We cannot prove a statement by proving its converse.
- (e) F. Although every statement is correct, the justification is incomplete. Without additional explanation the reader might wonder whether the proof means that  $x^2$  is always even and  $x + 1$  is always odd. One approach to a correct proof is to use the fact that  $x^2 + x$  is always even and that the product of an integer with an even integer is even. (Exercises 7(d) and 5(b).)

## 1.5 Basic Proof Methods II

1. (a) Suppose  $(G, *)$  is a not abelian.  
 $\vdots$   
 Thus,  $(G, *)$  is not a cyclic group.  
 Therefore, if  $(G, *)$  is a cyclic group, then  $(G, *)$  is abelian.
- (b) Suppose the determinant of  $\mathbf{B}$  is zero.  
 $\vdots$   
 Thus,  $\mathbf{B}$  is a singular matrix.  
 Therefore, if  $\mathbf{B}$  is a nonsingular matrix, then the determinant of  $\mathbf{B}$  is not zero.
- (c) Suppose the set of natural numbers is finite.  
 $\vdots$   
 Therefore  $Q$  (where  $Q$  is some proposition).  $\vdots$   
 Therefore  $\sim Q$ .  
 But  $Q$  and  $\sim Q$  is a contradiction.  
 Therefore, the set of natural numbers is not finite.
- (d) Suppose  $x$  is a real number other than 0. Then  $x$  has a multiplicative inverse, because  $x \cdot \frac{1}{x} = 1$ .  
 Suppose  $x$  has another multiplicative inverse  $z$ .  
 $\vdots$   
 Then  $P$ , where  $P$  is some proposition.  
 $\vdots$   
 Then  $\sim P$ .  
 Therefore  $P$  and  $\sim P$ , which is a contradiction.  
 We conclude that  $x$  has only one multiplicative inverse.
- (e) Part 1. Suppose the inverse of the function  $f$  from  $A$  to  $B$  is a function from  $B$  to  $A$ .  
 $\vdots$   
 Therefore  $f$  is one-to-one.  
 $\vdots$   
 Therefore  $f$  is onto  $B$ .  
 Part 2. Suppose  $f$  is one-to-one and onto  $B$ .  
 $\vdots$   
 Therefore the inverse of  $f$  is a function from  $B$  to  $A$ .
- (f) Part 1. Suppose  $A$  is compact.  
 $\vdots$   
 Therefore  $A$  is closed and bounded.



Part 2. Suppose  $A$  is closed and bounded.

$\vdots$

Therefore  $A$  is compact.

2. If  $\mathbf{A}$  and  $\mathbf{B}$  are invertible matrices, then  $\mathbf{AB}$  is invertible.

(a) Suppose  $\mathbf{AB}$  is not invertible.

$\vdots$

Thus,  $\mathbf{A}$  is not invertible or  $\mathbf{B}$  is not invertible.

Therefore, if  $\mathbf{A}$  and  $\mathbf{B}$  are invertible matrices, then  $\mathbf{AB}$  is invertible.

(b) Suppose  $A$  is not invertible or  $B$  is not invertible.

$\vdots$

Thus,  $\mathbf{AB}$  is not invertible.

Therefore, if  $\mathbf{AB}$  is invertible, then  $A$  and  $B$  are both invertible.

(c) Suppose both  $\mathbf{A}$  and  $\mathbf{B}$  are invertible, and  $\mathbf{AB}$  is not invertible.

$\vdots$

Therefore  $G$  (where  $G$  is some proposition).

$\vdots$

Therefore  $\sim G$ .

Hence  $G$  and  $\sim G$ , which is a contradiction.

Therefore, if  $\mathbf{A}$  and  $\mathbf{B}$  are invertible matrices, then  $\mathbf{AB}$  is invertible.

(d) Suppose  $\mathbf{AB}$  is invertible, and at least one of  $A$  or  $B$  is not invertible.

$\vdots$

Therefore  $G$ .

$\vdots$

Therefore  $\sim G$ .

Hence  $G$  and  $\sim G$ , which is a contradiction.

Therefore, if  $\mathbf{AB}$  is invertible, then  $\mathbf{A}$  and  $\mathbf{B}$  are both invertible.

(e) Part 1. Assume  $\mathbf{A}$  and  $\mathbf{B}$  are invertible.

$\vdots$

Therefore  $\mathbf{AB}$  is invertible.

Part 2. Assume  $\mathbf{AB}$  is invertible.

$\vdots$

Then  $\mathbf{A}$  and  $\mathbf{B}$  are invertible.

We conclude that  $\mathbf{A}$  and  $\mathbf{B}$  are invertible if and only if  $\mathbf{AB}$  is invertible.

3. (a) Suppose  $x + 1$  is even (not odd). Then  $x + 1 = 2k$  for some integer  $k$ . Then  $x = 2k - 1 = 2(k - 1) + 1$  and  $k - 1$  is an integer, so  $x$  is odd. Therefore if  $x$  is even, then  $x + 1$  is odd.

(b) Suppose  $x + 2$  is even (not odd). Then there is an integer  $m$  such that  $x + 2 = 2m$ , so  $x = 2m - 2 = 2(m - 1)$ . Since  $m - 1$  is an integer,  $x$  is even. Therefore if  $x$  is odd, then  $x + 2$  is odd.

(c) Suppose  $x$  is even. Then  $x = 2k$  for some integer  $k$ . Thus  $x^2 = (2k)^2 = 4k^2$  and  $k^2$  is an integer, so  $x^2$  is divisible by 4. Therefore if  $x^2$  is not divisible by 4, then  $x$  is odd.

(d) Suppose  $x$  is odd and  $y$  is odd. Then  $x = 2k + 1$  and  $y = 2m + 1$  for some integers  $k$  and  $m$ . Then  $2km + m + k$  is an integer and  $2(2km + m + k) + 1 =$

$4km + 2m + 2k + 1 = (2k + 1)(2m + 1) = xy$ , so  $xy$  is odd. Therefore if  $xy$  is even, then  $x$  or  $y$  is even.

- (e) Suppose it is not the case that either  $x$  and  $y$  are both odd or  $x$  and  $y$  are both even. Then one of  $x$  or  $y$  is even and the other is odd. We may assume that  $x$  is even and  $y$  is odd. (Otherwise, we could relabel the two integers.) Then  $x = 2k$  and  $y = 2m + 1$  for some integers  $k$  and  $m$ . Then  $x + y = 2k + 2m + 1 = 2(k + m) + 1$  and  $k + m$  is an integer, so  $x + y$  is odd. Therefore if  $x + y$  is even then  $x$  and  $y$  are both odd or both even.
  - (f) Suppose  $x$  and  $y$  are not both odd. Then either  $x$  or  $y$  is even (or both are even). We may assume  $x$  is even. Then  $x = 2m$  for some integer  $m$ . Thus  $xy = (2m)y = 2(my)$ , and  $my$  is an integer, so  $xy$  is even. Therefore if  $xy$  is odd then both  $x$  and  $y$  are odd.
  - (g) Suppose  $x$  is odd. Then  $x = 2m + 1$  for some integer  $m$ . Then  $x^2 - 1 = (2m + 1)^2 - 1 = 4m^2 + 4m = 4m(m + 1)$ . By a previous exercise,  $m(m + 1)$  is even, so  $m(m + 1) = 2k$  for some integer  $k$ . Thus  $x^2 - 1 = 4(2k) = 8k$ , so 8 divides  $x^2 - 1$ . Therefore if 8 does not divide  $x^2 - 1$ , then  $x$  is even.
  - (h) Assume  $x$  divides  $z$ . Then  $z = xk$  for some integer  $k$ . Thus  $yz = y(xk) = x(yk)$  and  $yk$  is an integer, so  $x$  divides  $yz$ . Therefore if  $x$  does not divide  $yz$ , then  $x$  does not divide  $z$ .
4. (a) Suppose  $x \geq 0$ . Then  $x + 2 > 0$ , and so the product  $x(x + 2) = x^2 + 2x \geq 0$ . Therefore if  $x^2 + 2x < 0$ , then  $x < 0$ .
- (b) Suppose  $x \leq 2$  or  $x \geq 3$ .  
 If  $x \leq 2$ , then  $x - 2 \leq 0$  and  $x - 3 \leq 0$ , so  $(x - 2)(x - 3) = x^2 - 5x + 6 \geq 0$ .  
 If  $x \geq 3$ , then  $x - 3 \geq 0$  and  $x - 2 \geq 1 > 0$ , so  $(x - 3)(x - 2) = x^2 - 5x + 6 \geq 0$ .  
 Therefore if  $x^2 - 5x + 6 < 0$ , then  $2 < x$  and  $x < 3$ .
- (c) Suppose  $x \leq 0$ . Since  $x^2 \geq 0$ ,  $x^2 + 1 > 0$ . Thus the product  $x(x^2 + 1) = x^3 + x \leq 0$ . Therefore if  $x^3 + x > 0$ , then  $x > 0$ .
5. (a) Suppose  $(-1, 5)$  and  $(5, 1)$  are both on a circle with center  $(2, 4)$ . Then the radius of the circle is  $\sqrt{(2 + 1)^2 + (4 - 5)^2} = \sqrt{10}$  and the radius of the circle is  $\sqrt{(2 - 5)^2 + (4 - 1)^2} = \sqrt{18}$ . This is impossible. Therefore  $(-1, 5)$  and  $(5, 1)$  are not both on the circle.
- (b) Suppose the circle has radius less than 5 and there is a point  $(a, b)$  on the circle and on the line  $y = x - 6$ . Then  $b = a - 6$  and  $(a - 2)^2 + (b - 4)^2 < 25$ , so  $(a - 2)^2 + (a - 10)^2 < 25$ . Then  $2a^2 - 24a + 79 < 0$ , or  $2(a - 6)^2 + 7 < 0$ . But  $2(a - 6)^2 + 7 \geq 7$ , so  $2(a - 6)^2 + 7 < 0$  is impossible. Therefore the circle does not intersect the line  $y = x - 6$ .
- (c) Suppose the point  $(0, 3)$  is not inside the circle, but  $(3, 1)$  is inside the circle. Then the distance from  $(2, 4)$  to  $(3, 1)$  is less than the radius and the distance from  $(2, 4)$  to  $(0, 3)$  is greater. Therefore  $(2 - 3)^2 + (4 - 1)^2 < (2 - 0)^2 + (4 - 3)^2$ . But  $1 + 9$  is *not* less than  $4 + 1$ . Therefore if  $(0, 3)$  is not inside the circle, then  $(3, 1)$  is not inside the circle.
6. (a) Let  $a$  and  $b$  be positive integers. Suppose  $a$  divides  $b$  and  $a > b$ . Then there is a natural number  $k$  such that  $b = ak$ . Since  $k$  is a natural number,  $k \geq 1$ . Thus  $b = ak \geq a \cdot 1 = a$ . Thus  $b \geq a$ . This contradicts the assumption that  $a > b$ . Therefore if  $a$  divides  $b$ , then  $a \leq b$ .
- (b) Let  $a$  and  $b$  be positive integers. Suppose  $ab$  is odd and that  $a$  or  $b$  is even. We may assume  $a$  is even. Then  $a = 2m$  for some integer  $m$ . Then  $ab = (2m)b = 2(mb)$ . Since  $mb$  is an integer,  $ab$  is even. Since a number cannot be both even and odd, this is a contradiction. Therefore if  $ab$  is odd, then  $a$  and  $b$  are both odd.

- (c) Suppose  $a$  is odd and  $a+1$  is not even. Then  $a+1$  is odd, so  $a+1 = 2k+1$  for some integer  $k$ . Thus  $a = 2k$ , so  $a$  is even. This contradicts the assumption that  $a$  is odd. Therefore if  $a$  is odd,  $a+1$  is even.
- (d) Suppose  $a-b$  is odd and  $a+b$  is even. Then  $a-b = 2k+1$  and  $a+b = 2m$  for some integers  $k$  and  $m$ . Then  $(a-b) + (a+b) = 2a = 2k+1 + 2m = 2(k+m) + 1$  is odd, but  $2a$  is even. This is a contradiction. Therefore if  $a-b$  is odd, then  $a+b$  is odd.
- (e) Let  $a, b$  be positive integers. Assume that  $a < b$  and  $ab < 3$ . Suppose that  $a \neq 1$ . Since  $a$  is a positive integer,  $a \geq 2$ . And since  $a < b$ ,  $b > 3$ . Therefore  $ab > 6$ . This contradicts the assumption that  $ab < 3$ .
7. (a) Let  $a, b$ , and  $c$  be positive integers. Then

$$\begin{aligned} a \text{ divides } b & \text{ iff } b = ak \text{ for some integer } k \\ & \text{ iff } bc = (ac)k \text{ for some integer } k \\ & \text{ iff } ac \text{ divides } bc. \end{aligned}$$

- (b) Let  $a$  and  $b$  be positive integers.
- Part 1. Suppose  $a = 2$  and  $b = 3$ . Then  $a+1 = 3$  divides  $b = 3$  and  $b = 3$  divides  $b+3 = 6$ .
- Part 2. Suppose  $a+1$  divides  $b$  and  $b$  divides  $b+3$ . Then  $b+3 = bk$  for some integer  $k$ , so  $3 = bk - b = b(k-1)$ . Therefore  $b$  divides 3, so  $b = 1$  or  $b = 3$ . Since  $a+1$  divides  $b$ ,  $a+1 \leq b$ . Thus  $b \neq 1$ , so  $b = 3$ . Since  $a+1 > 1$  and  $a+1$  divides 3,  $a+1 = 3$ . Thus  $a = 2$ .
- (c) Let  $a$  be a positive integer.
- Part 1. Suppose  $a$  is odd. Then  $a = 2k+1$  for some integer  $k$ , so  $a+1 = 2k+2 = 2(k+1)$ . Since  $k+1$  is an integer,  $a+1$  is even.
- Part 2. Suppose  $a+1$  is even. Then  $a+1 = 2m$  for some integer  $m$ , so  $a = 2m-1 = 2(m-1) + 1$ . Since  $m-1$  is an integer,  $a$  is odd.
- (d) Let  $a, b, c, d$  be positive integers.

$$\begin{aligned} a+b=c \text{ and } 2b-a=d & \text{ iff } a=b-c \text{ and } 2b-a=d \\ & \text{ iff } a=b-c \text{ and } 2b-(b-c)=d \\ & \text{ iff } a=b-c \text{ and } b+c=d. \end{aligned}$$

8. (a) Suppose  $m$  and  $n$  have the different parity. Then one is even and the other is odd. We may assume, without loss of generality, that  $m$  is even and  $n$  is odd. Then  $m = 2k$  and  $n = 2j+1$  for some integers  $k$  and  $j$ . Then  $m^2 - n^2 = (2k)^2 - (2j+1)^2 = 4k^2 - 4j^2 - 4j - 1 = 2(2k^2 - 2j^2 - 1) + 1$ , which is odd.
- (b) Suppose  $m^2 - n^2$  is odd.
- If  $m$  is even, then  $m^2$  is even. Therefore, since  $m^2 - n^2$  is odd and  $m^2$  is even,  $n^2 = m^2 - (m^2 - n^2)$  is odd. From  $n^2$  is odd, we conclude that  $n$  is odd.
- If  $m$  is odd, then  $m^2$  is odd. Therefore, since  $m^2 - n^2$  is odd and  $m^2$  is odd,  $n^2 = m^2 - (m^2 - n^2)$  is even. From  $n^2$  is even, we conclude that  $n$  is even.
- Hence, if  $m$  is even, then  $n$  is odd, and if  $m$  is odd, then  $n$  is even. Therefore,  $m$  and  $n$  have opposite parity.

9. Let  $n$  be a natural and suppose to the contrary that  $\frac{n}{n+1} \leq \frac{n}{n+2}$ . Then  $n(n+2) \leq n(n+1)$ , so  $n+2 \leq n+1$ , which is impossible. We conclude that  $\frac{n}{n+1} > \frac{n}{n+2}$ .
10. Assume  $\sqrt{5}$  is a rational number. Then  $\sqrt{5} = \frac{p}{q}$  where  $p$  and  $q$  are positive integers,  $q \neq 0$ , and  $p$  and  $q$  have no common factors. Then  $5 = \frac{p^2}{q^2}$ , so  $5q^2 = p^2$ . The prime factorization of  $q^2$  has an even number of 5's (twice as many as the factorization of  $q$ ), so  $5q^2$  has an odd number of 5's in its prime factorization. But  $5q^2 = p^2$  and  $p^2$  has an even number of 5's in its prime factorization. This is impossible. Therefore  $\sqrt{5}$  is irrational.
11. Suppose  $x, y, z$  are real numbers and  $0 < x < y < z < 1$ . Assume that the distances from  $x$  to  $y$  and from  $y$  to  $z$  are at least  $\frac{1}{2}$ . That is, assume  $|x - y| = y - x \geq \frac{1}{2}$  and  $|y - z| = z - y \geq \frac{1}{2}$ . The total distance from 0 to 1 is 1; that is  $(x - 0) + (y - x) + (z - y) + (1 - z) = 1$ . But  $x - 0 > 0$  and  $1 - z > 0$ , so  $(x - 0) + (y - x) + (z - y) + (1 - z) > 0 + \frac{1}{2} + \frac{1}{2} + 0 = 1$ . This is a contradiction. Therefore at least two of  $x, y, z$  are within  $\frac{1}{2}$  unit from one another.
12. (a) F. This is a proof of the converse of the statement, by contraposition.  
 (b) A.  
 (c) F. This seems to be a persuasive "proof" that the sum of two even integers is even, but it assumes that the sum of even numbers is even, which is what must be proved.  
 (d) C. Leaving out the assumption that  $a$  divides  $b$  and  $a$  divides  $c$  makes this proof confusing. If we change the first sentence to "Assume  $a$  divides both  $b$  and  $c$  and  $a$  does not divide  $b + c$ " then we have a correct proof by contradiction.

## 1.6 Proofs Involving Quantifiers

1. (a) Choose  $m = -3$  and  $n = 1$ . Then  $2m + 7n = 1$ .  
 (b) Choose  $m = 1$  and  $n = -1$ . Then  $15m + 12n = 3$ .  
 (c) Suppose  $m$  and  $n$  are integers and  $2m + 4n = 7$ . Then 2 divides  $2m$  and 2 divides  $4n$ , so 2 divides their sum  $2m + 4n$ . But 2 does not divide 7, so this is impossible.  
 (d) Suppose  $12m + 15n = 1$  for some integers  $m$  and  $n$ . Then 3 divides the left side but not the right side, which is impossible.  
 (e) Let  $t$  be an integer. Suppose there exist integers  $m$  and  $n$  such that  $15m + 16n = t$ . Let  $r$  and  $s$  be the integers  $5m$  and  $2n$ , respectively. Then  $3r + 8s = 15m + 16n = t$ .  
 (f) Suppose  $12m + 15n = 1$  for some integers  $m$  and  $n$ , and suppose further that  $m$  and  $n$  are not both positive. Then 3 divides the left side but not the right, which is impossible. Therefore, if there exist integers  $m$  and  $n$  such that  $12m + 15n = 1$ , then  $m$  and  $n$  are both positive.

**Alternative proof.** Let  $P$  be the statement "there exist integers  $m$  and  $n$  such that  $12m + 15n = 1$ ." By part (d),  $P$  is false. Therefore  $P$  implies  $m$  and  $n$  are both positive.

- (g) Let  $m$  be an odd integer. Suppose  $m = 4k + 1$  for some integer  $k$ . Then  $m + 2 = 4k + 3 = 4(k + 1) - 1 = 4j - 1$  where  $j$  is the integer  $k + 1$ .

- (h) Suppose  $m$  is an odd integer. Then  $m = 2n + 1$  for some integer  $n$ . Then  $m^2 = (2n + 1)^2 = 4n^2 + 4n + 1 = 4(n^2 + n) + 1$ . But  $n^2 + n = n(n + 1)$  is even (Recall Exercise 7(d) of Section 1.4), so  $n^2 + n = 2k$  for some integer  $k$ . Therefore  $m^2 = 4(2k) + 1 = 8k + 1$ .
- (i) Suppose  $m$  and  $n$  are odd integers. Assume that neither  $m$  nor  $n$  is of the form  $4j - 1$  for some integer  $j$ . Then  $m = 4j_1 + 1$  and  $n = 4j_2 + 1$ , for some integers  $j_1$  and  $j_2$ . (Recall the result from Section 1.4 that every odd integer has either the form  $4j + 1$  or the form  $4j - 1$ , for some integer  $j$ .) Thus  $mn = (4j_1 + 1)(4j_2 + 1) = 4(4j_1j_2 + j_1 + j_2) + 1$ . Then  $mn$  cannot be written in the form  $4k - 1$ , where  $k$  is an integer. Therefore, if  $mn$  is of the form  $4k - 1$ , then  $m$  or  $n$  is of the form  $4j - 1$ .
2. (a) Let  $a$ ,  $b$  and  $c$  be integers such that  $c$  divides  $a$  and  $c$  divides  $b$ . Let  $x$  and  $y$  be integers. Then there exist integers  $m$  and  $n$  such that  $a = cm$  and  $b = cn$ , so  $ax + by = (cm)x + (cn)y = c(mx + ny)$  and  $mx + ny$  is an integer, so  $c$  divides  $ax + by$ .
- Alternate proof.** Let  $a$ ,  $b$  and  $c$  be integers such that  $c$  divides  $a$  and  $c$  divides  $b$ . Let  $x$  and  $y$  be integers. By a previous exercise,  $c$  divides integer multiples of  $a$  and  $b$ , so  $c$  divides  $ax$  and  $by$ . Then (using another exercise)  $c$  divides their sum  $ax + by$ .
- (b) The proof involves repeated applications of the previous exercise, which says that  $a$  divides any sum of integer multiples of  $a$ .
- Let  $a$ ,  $b$  and  $c$  be integers such that  $a$  divides  $b - 1$  and  $a$  divides  $c - 1$ . Then by exercise 2(a)  $a$  divides  $(b - 1)(c - 1) = bc - b - c + 1$ . Then by Exercise 2(a),  $a$  divides  $(bc - b - c + 1) + (b - 1) = bc - c$ . Then by Exercise 2(a),  $a$  divides  $(bc - c) + (c - 1) = bc - 1$ .
- (c) Let  $a$  and  $b$  be integers such that  $b = ka$  for some integer  $k$ . Let  $n \in \mathbb{N}$ . Then  $b^n = (ka)^n = k^n a^n$ , so  $a^n$  divides  $b^n$ .
- (d) Let  $a$  and  $c$  be integers such that  $a$  is odd,  $c > 0$ ,  $c$  divides  $a$  and  $c$  divides  $a + 2$ .
- By Exercise 2(a),  $c$  divides  $(-1)a + 1(a + 2)$ , so  $c$  divides 2. The only divisors of 2 are 1 and 2. If  $c = 2$ , then 2 divides  $a$ , so  $a$  is even. But  $a$  is odd, so  $c = 1$ .
- (e) and suppose there exist integers  $m$  and  $n$  such that  $am + bn = 1$ . Also suppose that  $c$  divides  $a$  and  $c$  divides  $b$ . Then by part (a),  $c$  divides  $am + bn$ , so  $c$  divides 1. Thus  $c = \pm 1$ . Therefore if there exist integers  $m$  and  $n$  such that  $am + bn = 1$ , and  $c \neq \pm 1$ , then  $c$  does not divide  $a$  or  $c$  does not divide  $b$ .
3. Assume that every even natural number greater than 2 is the sum of two primes. Let  $n$  be an odd natural number greater than 5. Then  $n - 3$  is an even natural number greater than 2, so by the hypothesis it is the sum of two primes. Let  $p_1$  and  $p_2$  be primes such that  $n - 3 = p_1 + p_2$ . Then  $n = p_1 + p_2 + 3$ . Since  $p_1$ ,  $p_2$ , and 3 are primes,  $n$  is the sum of three primes.
4. (a) False.
- Counterexample: Let  $x = 41$ . Then  $x^2 + x + 41 = 41^2 + 41 + 41 = 41(41 + 1 + 1) = 41(43)$ , which is not prime.
- (b) True.
- Proof.** Let  $x$  be a real number. Then  $y = -x$  is a real number such that  $x + y = 0$ .

(c) False.

Counterexample: Let  $x = 2$  and  $y = 1$ . Then  $y^x = 1^2 = 1 \neq 2 = x$ .

(d) False.

Counterexample: Let  $a = 6$ ,  $b = 3$  and  $c = 2$ . Then  $a$  divides  $bc$ , but  $a$  doesn't divide  $b$  or  $c$ .

(e) True.

**Proof.** Suppose  $a, b, c, d, j$  and  $k$  are integers such that  $b - c = ka$  and  $c - d = ja$ . Then  $b - d = (b - c) + (c - d) = ka + ja = (k + j)a$ , so  $a$  divides  $b - d$ .

(f) False.

Counterexample: Let  $x = \frac{1}{2}$ . Then  $x^2 - x = -\frac{1}{4} < 0$ .

(g) False.

Counterexample: Let  $x = \frac{1}{2}$ . Then  $2^x = \sqrt{2} \approx 1.415 < 1.5 = x + 1$ .

(h) False.

Counterexample: Let  $x = 1$ . Suppose  $y$  is a positive real number less than  $x$ . Then  $0 < y < 1$ , and for  $z = 2$  (or any other positive real number)  $yz < z$ .

(i) True.

**Proof.** Let  $x$  be a positive real number and choose  $y = x$ . Then the statement  $[y < x \Rightarrow (\forall z)(yz \geq z)]$  is true.

**Alternative proof.** Let  $x$  be a positive real number, and let  $y = 1$ . Then if  $z$  is a positive real number,  $yz = z$ , so  $yz \geq z$ .

5. (a) Part 1. Suppose  $x$  is prime. Then by definition  $x$  is not 1 and there is no positive integer greater than 1 and less than or equal to  $\sqrt{x}$  that divides  $x$ . Part 2. Assume that  $x > 1$  and there is no positive integer greater than 1 and less than or equal to that divides  $x$ . Suppose  $x = mn$  for some natural numbers  $m$  and  $n$ , and  $m \leq n$ . By the hypothesis  $m = 1$  or  $m > \sqrt{x}$ . But  $m > \sqrt{x}$  implies that  $n > \sqrt{x}$ , from which we conclude that  $mn > x$ . Since this is impossible,  $m = 1$  and thus  $n = x$ . Therefore  $x$  is prime.
- (b) Suppose  $p$  is prime and  $p \neq 3$ . Then 3 does not divide  $p$ , so when  $p$  is divided by 3 the remainder is either 1 or 2. Thus, there is an integer  $k$  such that  $p = 3k + 1$  or there is an integer  $k$  such that  $p = 3k + 2$ .  
 If  $p = 3k + 1$ , then  $p^2 + 2 = (3k + 1)^2 + 2 = (9k^2 + 6k + 1) + 2 = 9k^2 + 6k + 3 = 3(3k^2 + 2k + 1)$  so  $p^2 + 2$  is divisible by 3.  
 If  $p = 3k + 2$ , then  $p^2 + 2 = (3k + 2)^2 + 2 = (9k^2 + 12k + 4) + 2 = 9k^2 + 12k + 6 = 3(3k^2 + 4k + 2)$  so  $p^2 + 2$  is divisible by 3.
6. (a) Let  $n$  be a natural number. Then  $n \geq 1$ , so  $1 = \frac{n}{n} \geq \frac{1}{n}$ .
- (b) Choose  $M = 10$ . Let  $n$  be a natural number greater than  $m$ . Then  $\frac{1}{n} < \frac{1}{M} = 0.1 < 0.13$ .
- (c) Let  $n$  be a natural number. Then both  $2n$  and  $2n + 1$  are natural numbers. Let  $M = 2n + 1$ . Then  $M$  is a natural number greater than  $2n$ .
- (d) Let  $M = 2$ . Now if  $n$  is a natural number, then by part (a)  $\frac{1}{n} \leq 1 < 2$ .
- (e) Suppose there is a largest natural number  $K$ . Then  $K + 1$  is a natural number and  $K + 1 > K$ . This is a contradiction.
- (f) Let  $\epsilon$  be a positive real number. Then  $\frac{\epsilon}{2}$  is a smaller positive real number. Therefore, for every positive real number there is a smaller positive real number.

- (g) Let  $\epsilon > 0$  be a real number. Then  $\frac{1}{\epsilon}$  is a positive real number and so has a decimal expression as an integer part plus a decimal part. Let  $M$  be the integer part of  $\frac{1}{\epsilon}$  plus 1. Then  $M$  is an integer and  $M > \frac{1}{\epsilon}$ .  
To prove for all natural numbers  $n > M$  that  $\frac{1}{n} < \epsilon$ , let  $n$  be a natural number and assume that  $n > M$ . Since  $M > \frac{1}{\epsilon}$ , we have  $n > \frac{1}{\epsilon}$ . Thus  $\frac{1}{n} < \epsilon$ . Therefore, for every real number  $\epsilon > 0$ , there is a natural number  $M$  such that for all natural numbers  $n > M$ ,  $\frac{1}{n} < \epsilon$ .
- (h) Let  $\epsilon > 0$ . By part (g), there is  $M$  such that  $\frac{1}{n} < \epsilon$  for all  $n > M$ . Now if  $m > n > M$ , then  $\frac{1}{n} - \frac{1}{m} < \frac{1}{n} < \epsilon$ .
- (i) Let  $K$  be 10. Suppose  $r > K$ . Then  $r > 10$ , so  $r^2 > 100$ . Therefore  $\frac{1}{r^2} < 0.01$ .
- (j) Let  $L = -15$  and  $G = -1$ . Suppose  $L < x < G$ . Then  $-1.5 < x < -1$ , so  $15 > -x > 1$ . Then  $30 > -2x > 2$  and  $40 > 10 - 2x > 12$ .
- (k) Let  $M$  be 51. Then  $M$  is an odd integer. Suppose  $r$  is a real number and  $r > M$ . Then  $r > 51$ , so  $r > 50$ . Then  $2r > 100$ , so  $\frac{1}{2r} < \frac{1}{100}$ .
- (l) Let  $x$  be a natural number. Choose  $k = -4x + 50$ . Then  $k$  is an integer and  $k < -3.3x + 50$ , so  $3.3x + k < 50$ .
- (m) Let  $x$  be 99 and  $y = 28$ . Then  $x + y = 127 < 128$ . Suppose  $r > x$  and  $s > y$ . Then  $r - 50 > 49$  and  $s - 20 > 8$ . Therefore  $(r - 50)(s - 20) > 392 > 390$ .
- (n) Let  $x$  and  $y$  be positive real numbers such that  $x < y$ . Then  $y - x$  is positive, and  $\frac{1}{y-x}$  is a positive real number. Choose  $M$  to be a natural number larger than  $\frac{1}{y-x}$ . Suppose  $n$  is a natural number and  $n > M$ . Then  $n > \frac{1}{y-x}$ , so  $y - x > \frac{1}{n}$ . That is,  $\frac{1}{n} < y - x$ .
7. (a) F. The false statement referred to is not a denial of the claim.  
(b) C. Uniqueness has not been shown.  
(c) F. Listing numerous examples does not constitute a proof.  
(d) A.  
(e) F. The proof is correct for Case 2. However, giving examples for Cases 1 and 3 does not prove that the statement is true for all  $x$  in those cases.  
(f) A. Terse, but correct.  
(g) A. A proof by contraposition would be more natural.  
(h) F. The “proof” shows that the converse of the claim is true.  
(i) C. The number  $\frac{1}{2\epsilon}$  may not be a natural number. To correct this error, choose  $K$  to be a natural number greater than  $\frac{1}{2\epsilon}$ .  
(j) A.

## 1.7 Additional Examples of Proofs

1. (a) **Proof.** We work both forwards and backwards: From the hypothesis that  $n$  is odd we can deduce that  $3n$  is even, from which we can deduce that  $3n + 1$  is odd. We could reach the conclusion that  $2n + 8$  is divisible by 4 if we knew that 4 divides  $2n$  (since 8 is divisible by 4). In turn, the statement  $2n$  is divisible by 4 may be derived from the statement that  $n$  is divisible by 2. We combine these steps in the proper order to create the proof.  
Suppose  $n$  is an integer and is odd. Therefore  $3n$  is even,  $3n + 1$  which implies that  $n$  is even. We are now using properties of even and odd integers that we proved earlier, without referencing specific examples or exercises. Since  $n$  is even,  $n$  is divisible by 2. Therefore  $2n$  is divisible by 4. Finally since 8 is also divisible by 4,  $2n + 8$  is divisible by 4.

- (b) **Proof.** Let  $a$  be a real number,  $a \neq 3$ . The key to the proof is to use the  $a = 3$  idea of “solution” and then work with the resulting equation.

Assume that  $a$  is a solution to  $x^2 - x - 6 = 0$ .

Then  $a$  makes the equation true by the definition of a solution to an equation.

Thus  $a^2 - a - 6 = (a - 3)(a + 2) = 0$ .

Then  $a + 2 = 0$ , because  $a - 3 \neq 0$ .

Then  $(a^2 + 1)(a + 2) = a^3 + 2a^2 + a + 3 \neq 0$ .

Therefore  $a$  is a solution to  $x^3 + 2x^2 + x + 3 = 0$ .

- (c) **Proof.** Assume that  $a \neq 3$ . Observe that in the proof above, each step implies its predecessor. Thus we can modify the given proof to create an iff proof.

$a$  is a solution to  $x^2 - x - 6 = 0$

iff  $a^2 - a - 6 = (a - 3)(a + 2) = 0$ .

iff  $a + 2 = 0$ .

Because  $a - 3 \neq 0$ .

iff  $(a^2 + 1)(a + 2) = a^3 + 2a^2 + a + 3 = 0$ .

Because  $a^2 + 1 \neq 0$ .

iff  $a$  is a solution to  $x^3 + 2x^2 + x + 3 = 0$ .

- (d) **Proof.** Suppose  $x^2 = 2x + 15$  and  $x > 2$ . Then  $(x - 5)(x + 3) = 0$ . Since  $x > 2$ ,  $x$  must be 5. Then  $x - 4$  and  $x - 3$  are positive, so  $(x - 4)/(x - 3) > 0$ .

- (e) **Proof.** Let  $x$  and  $y$  be real numbers. The statement has the form  $P \Rightarrow (Q \vee R)$ , so it might be proved by assuming  $P$  and  $\sim Q$  and deducing  $R$ . In this case a proof by contrapositive works well.

Assume that neither  $x$  nor  $y$  is irrational. Then both  $x$  and  $y$  are rational, so they can be written in the form  $x = \frac{p}{q}$  and  $y = \frac{r}{s}$  where  $p, q, r$ , and  $s$  are integers,  $q \neq 0$ , and  $s \neq 0$ . Therefore,  $x + y = \frac{p}{q} + \frac{r}{s} = \frac{ps+q}{qs}$ . Since  $ps + rq$  and  $qs$  are integers and  $qs \neq 0$ ,  $x + y$  is a rational number. We have shown that if  $x$  and  $y$  are rational, then  $x + y$  is rational. We conclude that if  $x + y$  is irrational, then either  $x$  or  $y$  is irrational.

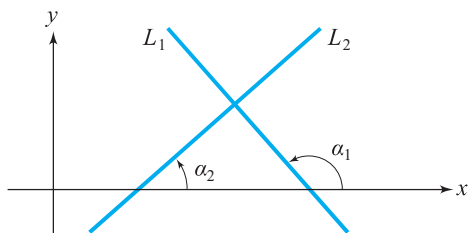
- (f) **Proof.** If we let  $S$  be the set of all nonvertical lines in the  $xy$ -plane, we can simplify the symbolic form of the theorem as follows:  $(\forall L_1 \in S)(\forall L_2 \in S)$   
 $(L_1 \text{ and } L_2 \text{ are perpendicular} \Rightarrow$   
 $(\text{slope of } L_1) \cdot (\text{slope of } L_2) = -1) >$ .

Let  $L_1$  and  $L_2$  be nonvertical lines. Suppose  $L_1$  and  $L_2$  are perpendicular. We now use the fact that the slope of a nonvertical line is  $\tan(\alpha)$ , where  $\alpha$  is the angle of inclination of the line. Let  $\alpha_1$  and  $\alpha_2$  be the angles of inclinations of  $L_1$  and  $L_2$ , respectively. See the figure below. We may assume that  $\alpha_1 > \alpha_2$ . We can make this assumption because the two lines are arbitrary; if  $\alpha_1 < \alpha_2$  simply interchange the labels of the lines. Since  $L_1$  and  $L_2$  are perpendicular,  $\alpha_1 = \alpha_2 + \frac{\pi}{2}$ . Therefore,

$$\tan(\alpha_1) = \tan(\alpha_2 + \frac{\pi}{2}) = -\cot(\alpha_2) = -\frac{1}{\tan(\alpha_2)}.$$

We use trigonometric identities to rewrite  $\tan(\alpha_1)$ . Thus,  $\tan(\alpha_1) \cdot \tan(\alpha_2) = -1$ . Since  $\tan(\alpha_1)$  is the slope of  $L_1$  and  $\tan(\alpha_2)$  is the slope of  $L_2$ , the product of the slopes is  $-1$ .





- (g) **Proof.** This is a “non-existence” proof. We could restate the result as “Every point inside the circle is not on the line” and begin a direct proof by assuming that  $(x, y)$  is a point inside the circle. We would then have to prove that  $(x, y)$  is not on the line. In this instance, a better approach is to use a proof by contradiction. The statement has the form  $\sim (\exists x)(\exists y)((x, y)$  is inside the circle  $\wedge (x, y)$  is on the line).

Suppose there is a point  $(a, b)$  that is inside the circle and on the line. Then  $(a - 3)^2 + b^2 < 6$  and  $b = a + 1$ . We now have two expressions to use. Therefore,

$$\begin{aligned} (a - 3)^2 + (a + 1)^2 &< 6 \\ 2a^2 - 4a + 10 &< 6 \\ a^2 - 2a + 5 &< 3 \\ a^2 - 2a + 1 &< -1 \\ (a - 1)^2 &< -1. \end{aligned}$$

This is a contradiction since  $(a - 1)^2 = 0$ . Thus, no point inside the circle is on the line.

- (h) **Proof.** Proofs that verify equalities or inequalities containing absolute value expressions usually involve cases, because of the two-part definition of  $|x|$ . The two cases are  $x - 2 \geq 0$  and  $x - 2 < 0$ . The proof in each case is discovered by working backwards from the desired conclusion. The key steps are to note that, in the first case, if  $x \geq 2$ , then  $-6 \leq x$ , and, in the second case, that if  $x \geq 1$ , then  $\frac{6}{7} \leq x$ .

Let  $x$  be a real number greater than 1.

Case 1. Suppose  $x - 2 \geq 0$ . Then  $|x - 2| = x - 2$ . Since  $x \geq 2$ ,

$$\begin{aligned} -6 &\leq x \\ 3x - 6 &\leq 4x \\ \frac{3(x - 2)}{x} &\leq 4. \end{aligned} \quad \text{Remember that } x \text{ is positive.}$$

Therefore,  $\frac{3|x - 2|}{x} \leq 4$ .

Case 2. Suppose  $x - 2 < 0$ . Then  $|x - 2| = -(x - 2)$ . By hypothesis,  $x \geq 1$ . Therefore,

$$\begin{aligned} \frac{6}{7} &\leq x \\ 6 &\leq 7x \\ 6 - 3x &\leq 4x \\ 3[-(x - 2)] &\leq 4x \end{aligned}$$

$$\frac{3[-(x-2)]}{x} \leq 4. \quad \text{Remember that } x \text{ is positive.}$$

Therefore,  $\frac{3|x-2|}{x} \leq 4.$

2. (a) Let  $n$  be an integer. Then  $n$  is either even or odd. If  $n$  is even then  $n = 2k$  for some integer  $k$ , so that  $5n^2 + 3n + 4 = 5(2k)^2 + 3(2k) + 4$ , which is twice the integer  $10k^2 + 3n + 2$ . If  $n$  is odd then  $n = 2k + 1$  for some integer  $k$ , so that  $5n^2 + 3n + 4 = 5(2k + 1)^2 + 3(2k + 1) + 4$ , which is twice the integer  $10k^2 + 13n + 6$ . In either case,  $5n^2 + 3n + 4$  is even.
- (b) Let  $n$  be an odd integer. Then  $n = 2k + 1$  for some integer  $k$ , so  $2n^2 + 3n + 4 = 2(2k + 1)^2 + 3(2k + 1) + 4 = 2(4k^2 + 7k + 4) + 1$ . Since  $4k^2 + 7k + 4$  is an integer,  $2n^2 + 3n + 4$  is odd.
- (c) Let  $x$  be the smallest of five consecutive integers. Then the sum is  $x + (x + 1) + (x + 2) + (x + 3) + (x + 4) = 5x + 10 = 5(x + 2)$ . Since  $x + 2$  is an integer, the sum is divisible by 5.
- (d) Let  $L_1$  and  $L_2$  be two nonvertical lines such that the product of their slopes is  $-1$ . Let  $\alpha_1$  and  $\alpha_2$  be the inclination angles of  $L_1$  and  $L_2$  respectively. Since neither line is vertical, the slope of  $L_1$  is  $\tan(\alpha_1)$  and the slope of  $L_2$  is  $\tan(\alpha_2)$ . Since  $\tan(\alpha_1)\tan(\alpha_2) = -1$ , exactly one of these two must be positive and neither can be 0. Suppose without loss of generality that  $\tan(\alpha_2) > 0$ . Then  $0 < \alpha_2 < \frac{\pi}{2}$ . Now  $\tan(\alpha_1) = \frac{-1}{\tan(\alpha_2)} = -\cot(\alpha_2) = \tan(\alpha_2 + \frac{\pi}{2})$ , and since both  $\alpha_1$  and  $\alpha_2 + \frac{\pi}{2}$  are between  $\frac{\pi}{2}$  and  $\pi$ , we must have that  $\alpha_1 = \alpha_2 + \frac{\pi}{2}$ . Therefore  $L_1$  and  $L_2$  are perpendicular.
- (e) Let  $n$  be an integer. Then  $n^3 - n = n(n^2 - 1) = n(n + 1)(n - 1)$  is the product of three consecutive integers. By previous results, if  $n$  is even, then  $n + 1$  is odd and if  $n$  is odd, then  $n + 1$  is even. Therefore, either  $n$  or  $n + 1$  is divisible by 2. By the Division Algorithm, the remainder when  $n$  is divided by 3 will be 0, 1, or 2.
- Case 1. If the remainder is 0, then  $n$  is divisible by 3.
- Case 2. If the remainder is 1,  $n = 3k + 1$  for some integer  $k$ . Then  $n - 1 = 3k$  is divisible by 3.
- Case 3. If the remainder is 2,  $n = 3k + 2$  for some integer  $k$ . Then  $n + 1 = 3k + 2 + 1 = 3k + 3$  is divisible by 3.
- In all cases, one of  $n$ ,  $n - 1$ ,  $n + 1$  has a factor of 3, and either  $n$  or  $n + 1$  has a factor of 2. Therefore,  $n^3 - n$  has a factor of  $2 \cdot 3$  and is divisible by 6.
- (f) Let  $n$  be an integer. Then  $(n^3 - n)(n + 2) = n(n + 1)(n - 1)(n + 2)$  is the product of four consecutive integers. From part (e)  $n^3 - n$  is divisible by 6 and hence divisible by 3.
- By previous results, if  $n$  is even, then  $n + 1$  is odd and if  $n$  is odd, then  $n + 1$  is even. Therefore, either  $n$  or  $n + 1$  is divisible by 2. If  $n$  is divisible by 2, then so is  $n + 2$ . If  $n + 1$  is divisible by 2, then so is  $n + 1 - 2 = n - 1$ . Thus either both  $n$  and  $n + 2$  are each divisible by 2 or both  $n - 1$  and  $n + 1$  are each divisible by 2.
- In all cases,  $n^3 - n$  has a factor of 3, and two terms (either  $n$  and  $n + 2$ , or  $n - 1$  and  $n + 1$ ) each have a factor of 2. Therefore,  $(n^3 - n)(n + 2)$  has two factors of 2 and one factor of 3 and therefore is divisible by  $2 \cdot 2 \cdot 3 = 12$ .
3. (a) Suppose the line  $2x + ky = 3k$  has slope  $\frac{1}{3}$ . In slope-intercept form the line has equation  $y = -\frac{2}{k}x + 3$ , so  $-\frac{2}{k} = \frac{1}{3}$ . Thus  $k = -6$ . Therefore if  $k \neq -6$ , then  $L$  does not have slope  $\frac{1}{3}$ .

- (b) Suppose for some real number  $k$  that  $L$  is parallel to the  $x$ -axis. Then  $L$  has slope 0, so  $\frac{-2}{k} = 0$ . This is impossible. Therefore  $L$  is not parallel to the  $x$ -axis.
- (c)  $L$  passes through  $(1, 4)$  iff  $2(1) + k(4) = 3k$ , and this happens iff  $k = -2$ .
4. (a) Suppose  $x$  is rational. Suppose that  $x + y$  is also rational. Then there exist integers  $p$  and  $q, q \neq 0$  such that  $x = \frac{p}{q}$  and integers  $r$  and  $s, s \neq 0$ , such that  $x + y = \frac{r}{s}$ . Then  $y = (x + y) - x = \frac{r}{s} - \frac{p}{q} = \frac{rq - ps}{sq}$ . Since  $rq - ps$  and  $sq$  are integers and  $sq \neq 0$ ,  $y$  is rational. Therefore if  $x$  is rational and  $y$  is irrational, then  $x + y$  is irrational.
- (b) Let  $x = \pi$  and  $y = -\pi$ . Then  $x$  and  $y$  are irrational, and  $x + y$  is irrational.
- (c) Let  $z$  be a rational number. By part (a),  $z + \pi$  is irrational. Let  $x = z + \pi$  and  $y = -\pi$ . Then  $x + y = z$ .
- (d) Let  $z$  be a rational number and  $x$  be an irrational number. Then  $-x$  is irrational. Let  $y = z - x$ . Then  $x + y = x + (z - x) = z$  and  $z - x$  is irrational by part (a), so there exists an irrational number  $y$  such that  $x + y = z$ .
- Suppose there is an irrational number  $w$  such that  $x + w = z$ . Then  $w = z - x = y$ . Therefore the irrational number  $y$  such that  $x + y = z$  is unique.
5. (a) Assume  $(x, y)$  is on the given circle and  $y \neq 0$ . Then  $x^2 + y^2 = r^2$ , so  $-y^2 = x^2 - r^2 = (x + r)(x - r)$  and so  $\frac{y}{x - r} = -\frac{x + r}{y}$ .
- Thus the slope of the line passing through  $(x, y)$  and  $(r, 0)$  is the negative reciprocal of the slope of the line that passes through  $(x, y)$  and  $(-r, 0)$ . Therefore the lines are perpendicular.
- Note that if  $y = 0$ , then  $x = \pm r$ , so the points for which this argument does not apply are  $(r, 0)$  and  $(-r, 0)$ . If  $(x, y) = (r, 0)$ , then there are many lines passing through  $(x, y)$  and  $(r, 0)$ , only one of which is perpendicular to the line through  $(x, y)$  and  $(-r, 0)$ .
- (b) First observe that if  $y = 0$ , then the points  $(x, y)$ ,  $(r, 0)$ , and  $(-r, 0)$  are all on the  $x$ -axis, so the line through  $(x, y)$  and  $(r, 0)$  is not perpendicular to the line through  $(x, y)$  and  $(-r, 0)$ .
- Suppose  $y \neq 0$  and the two lines are perpendicular. Then the slopes of the lines are negative reciprocals, so  $\frac{y}{x - r} = -\frac{x + r}{y}$ . Thus  $-y^2 = (x + r)(x - r) = x^2 - r^2$  and so  $x^2 + y^2 = r^2$ . But the fact that  $(x, y)$  lies inside the circle requires that  $x^2 + y^2 < r^2$ , so this is a contradiction.
6. (a) Then  $y_0 = 6 - x_0$ . The distance between  $(x_0, y_0)$  and  $(-3, 1)$  is

$$\begin{aligned}
 D &= \sqrt{(-3 - x_0)^2 + (1 - y_0)^2} \\
 &= \sqrt{(-3 - x_0)^2 + [1 - (6 - x_0)]^2} \\
 &= \sqrt{(-3 - x_0)^2 + (x_0 - 5)^2} \\
 &= \sqrt{2x_0^2 - 4x_0 + 34} \\
 &= \sqrt{2(x_0^2 - 2x_0 + 1) + 32} \\
 &= \sqrt{[2(x_0 - 1)^2 + 16] + 16} \\
 &> \sqrt{16} \quad \text{since } [2(x_0 - 1)^2 + 16] > 0
 \end{aligned}$$

$$= 4.$$

Since the distance from  $(-3, 1)$  to  $(x_0, y_0)$  is greater than 4, the point  $(x_0, y_0)$  is outside the circle.

- (b) 269 is such a number.
- (c) By the Extreme Value Theorem, if  $f$  does not have a maximum value on  $[5, 7]$ , then  $f$  is not continuous. And if  $f$  is not continuous on  $[5, 7]$ , then  $f$  is not differentiable on  $[5, 7]$ .
- (d) Suppose to the contrary that there are real numbers  $a < b$  that satisfy this equation. Since  $f(x) = x^3 + 6x - 1$  is continuous and differentiable everywhere, it is certainly continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Thus Rolle's Theorem asserts the existence of a number  $c \in (a, b)$  such that  $f'(c) = 3c^2 + 6 = 0$ , which is impossible since  $3c^2 \geq 0$ .
7. (a) (This proof is motivated by working backward from two desired inequalities – one for each possible value of  $|2x - 1|$ .)  
 Let  $x$  be a nonnegative real number.  
 Case 1:  $x \geq \frac{1}{2}$ . In this case  $2x - 1 \geq 0$ , so  $|2x - 1| = 2x - 1$ . Since  $-1 \leq 2$ , we have  $2x - 1 \leq 2x + 2 = 2(x + 1)$ . Since  $x + 1$  is positive, we have  $\frac{2x-1}{x+1} \leq 2$ . Thus  $\frac{|2x-1|}{x+1} \leq 2$ .  
 Case 2:  $0 < x < \frac{1}{2}$ . In this case  $2x - 1 < 0$ , so  $|2x - 1| = 1 - 2x$ . Since  $x > 0$ ,  $-1 < 4x$ . Thus  $1 - 2x < 2 + 2x = 2(1 + x)$ . Since  $1 + x$  is positive, we have  $\frac{1-2x}{1+x} < 2$ . Therefore  $\frac{|2x-1|}{x+1} < 2$ .  
 (b) If  $-2 < x < 1$ , then  $(x - 1) < 0$ ,  $(x + 2) > 0$ ,  $(x - 3) < 0$  and  $(x + 4) > 0$ , so  $\frac{(x-1)(x+2)}{(x-3)(x+4)} > 0$ .  
 Now if  $x > 3$ , then all of the factors are positive, so  $\frac{(x-1)(x+2)}{(x-3)(x+4)} > 0$ .
8. (a) Suppose  $(x, y)$  is inside the first circle. Then from the distance formula,  $|x - 3|^2 + |y - 2|^2 < 4$ . Therefore  $|x - 3|^2 < 4$  and  $|y - 2|^2 < 4$ . It follows that  $|x - 3| < 2$  and  $|y - 2| < 2$ , so  $-2 < x - 3 < 2$  and thus  $1 < x < 5$  and  $0 < y < 4$ . Therefore  $x^2 < 25$  and  $y^2 < 16$ , so  $x^2 + y^2 < 41$ . Thus  $(x, y)$  is inside the second circle.  
 (b) Suppose  $(x, y)$  is inside the circle. Then as in part (a),  $|x - 3| < 2$  and  $|y - 2| < 2$ , so in particular,  $x - 3 < 2$  and  $-2 < y - 2$ . Therefore  $x - 6 < -1 < 0 < y < 3y$ .  
 (c) The statement is false. The point  $(2, 3)$  is inside the circle  $(x-3)^2 + (y-2)^2 = 4$ , since  $(2-3)^2 + (3-2)^2 = 2 < 4$ . But  $(2, 3)$  is not inside  $(x-5)^2 + (y+1)^2 = 25$ , since  $(2-5)^2 + (3+1)^2 = 25 \not< 25$ .
9. (a)  $310 = 8(38) + 6$ . The quotient is 38 and remainder is 6.  
 (b)  $36 = 5(7) + 1$ . The quotient is 7 and remainder is 1.  
 (c)  $36 = -5(-7) + 1$ . The quotient is -7 and remainder is 1.  
 (d)  $-36 = 5(-8) + 4$ . The quotient is -8 and remainder is 4.  
 (e)  $44 = 7(6) + 2$ . The quotient is 6 and remainder is 2.  
 (f)  $-52 = -8(7) + 4$ . The quotient is 7 and remainder is 4.
10. (a) Suppose  $a$  and  $b$  are integers,  $a > b$ , and  $b \geq 0$ . Then  $b = a(0) + b$ , so when  $b$  is divided by  $a$ , the quotient is 0.

- (b) Suppose  $a$  and  $b$  are integers,  $a > b$ , and the quotient is 0 when  $b$  is divided by  $a$ . Then  $b = a(0) + r$ , where the remainder  $r$  is  $\geq 0$ . Then  $r = b$ , so  $b \geq 0$ .
11. (a)  $2, 1, -1, -2 \quad \gcd(8, 310) = 2$   
 (b)  $1, -1 \quad \gcd(-5, 36) = 1$   
 (c)  $18, 9, 6, 3, 2, 1, -1, -2, -3, -6, -9, -18 \gcd(18, -54) = 18$   
 (d)  $4, 2, 1, -1, -2, -4 \quad \gcd(-8, -52) = 4$
12. (a)  $2 = (2)12 + (-1)22$  and  $2 = (-9)12 + 5(22)$   
 (b)  $-4 = (7)12 + (-4)22$  and  $-4 = (-4)12 + 2(22)$   
 (c) The set of all linear combinations of 12 and 22 is the set of even integers.
13. (a)  $\gcd(13, 15) = 1. \quad 1 = (7)13 + (-6)15$   
 (b)  $\gcd(26, 32) = 2. \quad 2 = (5)26 + (-4)32$   
 (c)  $\gcd(9, 30) = 3. \quad 3 = (7)9 + (-2)30$
14. (a) Let  $a, b$ , and  $c$  be natural numbers and  $\gcd(a, b) = d$ . Suppose  $c$  divides  $a$  and  $c$  divides  $b$ . By Theorem 1.7.3,  $d$  is a linear combination of  $a$  and  $b$ . By Theorem 1.7.1,  $c$  divides every linear combination of  $a$  and  $b$ . Therefore  $c$  divides  $d$ .
- (b) Let  $a$  and  $b$  be natural numbers and  $\gcd(a, b) = d$ .
- Suppose  $a$  divides  $b$ . Then  $a$  is a common divisor of  $a$  and  $b$ . No number larger than  $a$  divides  $a$ , so  $a$  is the largest common divisor. Thus  $a = d$ .
  - Suppose  $a = d$ . Then  $a$  is a common divisor of  $a$  and  $b$ , so  $a$  divides  $b$ .
- (c) Let  $a, b$ , and  $c$  be natural numbers and  $\gcd(a, b) = d$ . Suppose  $a$  divides  $bc$  and  $d = 1$ . By Theorem 1.7.3,  $d$  is the smallest positive linear combination of  $a$  and  $b$ . Therefore there exist integers  $s$  and  $t$  such that  $as + bt = 1$ . Then  $acs + bct = c$ . Since  $a$  divides  $acs$  and  $a$  divides  $bct$ ,  $a$  divides their sum. Thus  $a$  divides  $c$ .
- (d) Let  $a, b$ , and  $c$  be natural numbers and  $\gcd(a, b) = d$ . Suppose  $c$  divides  $a$  and  $b$ . Since  $\gcd(a, b) = d$ ,  $c$  divides  $d$  (by part (a)) and there are integers  $s$  and  $t$  such that  $as + bt = d$ . Then

$$\left(\frac{c}{c}\right)as + \left(\frac{c}{c}\right)bt = d.$$

Therefore

$$\frac{a}{c}s + \frac{b}{c}t = \frac{d}{c}.$$

Since  $\frac{d}{c}$  is a linear combination of  $\frac{a}{c}$  and  $\frac{b}{c}$ , by Theorem 1.7.1  $\gcd(\frac{a}{c}, \frac{b}{c})$  divides  $\frac{d}{c}$ . Hence

$$\frac{d}{c} \geq \gcd\left(\frac{a}{c}, \frac{b}{c}\right).$$

By Theorem 1.7.3 there exist integers  $p$  and  $q$  such that

$$\frac{a}{c}p + \frac{b}{c}q = \gcd\left(\frac{a}{c}, \frac{b}{c}\right).$$

Therefore

$$ap + bq = c \cdot \gcd\left(\frac{a}{c}, \frac{b}{c}\right).$$

Since

$$c \cdot \gcd\left(\frac{a}{c}, \frac{b}{c}\right)$$

is a linear combination of  $a$  and  $b$ ,  $d$  divides

$$c \cdot \gcd\left(\frac{a}{c}, \frac{b}{c}\right).$$

Thus  $\frac{d}{c}$  divides

$$\gcd\left(\frac{a}{c}, \frac{b}{c}\right).$$

Hence

$$\frac{d}{c} \leq \gcd\left(\frac{a}{c}, \frac{b}{c}\right).$$

We conclude that

$$\frac{d}{c} = \gcd\left(\frac{a}{c}, \frac{b}{c}\right).$$

- (e) Let  $a$  and  $b$  be natural numbers and  $\gcd(a, b) = d$ . Let  $n$  be a natural number. Since  $\gcd(a, b) = d$ , there are integers  $s$  and  $t$  such that  $as + bt = d$ . Then  $n(as + bt) = nd$ . Therefore  $(an)s + (bn)t = dn$ . Since  $dn$  is a linear combination of  $an$  and  $bn$ , by Theorem 1.7.1  $dn \geq \gcd(an, bn)$ .

By Theorem 1.7.3 there exist integers  $p$  and  $q$  such that  $(an)p + (bn)q = \gcd(an, bn)$ . Therefore  $ap + bq = \frac{1}{n}\gcd(an, bn)$ . Since  $\frac{1}{n}\gcd(an, bn)$  is a linear combination of  $a$  and  $b$ ,  $d$  divides  $\frac{1}{n}\gcd(an, bn)$ . Thus,  $dn$  divides  $\gcd(an, bn)$ , which implies  $dn \leq \gcd(an, bn)$ .

We conclude that  $dn \equiv \gcd(an, bn)$ .

15. 3, 6, and 10 are relatively prime to 7.  
10 is relatively prime to 21.  
None is relatively prime to 30.
16. (a) Suppose  $p$  is prime and  $a$  is any natural number. The only divisors of  $p$  are 1 and  $p$ , and  $\gcd(p, a)$  divides  $p$ , so  $\gcd(p, a) = 1$  or  $p$ .
  - i. Assume  $\gcd(p, a) = p$ . Then  $p$  divides  $a$  by definition of  $\gcd$ .
  - ii. Suppose  $p$  divides  $a$ . Then  $p$  is a common divisor of  $p$  and  $a$ . Since  $p$  is the largest divisor of  $p$  it is the largest common divisor of  $p$  and  $a$ , so  $\gcd(p, a) = p$ .
- (b) Suppose  $p$  is prime and  $a$  is a natural number.
  - i. Suppose  $\gcd(p, a) = 1$ . Then  $p$  does not divide  $a$ , because otherwise  $p$  would be a divisor of both  $p$  and  $a$  that is larger than 1.
  - ii. Suppose  $p$  does not divide  $a$ . Then the only common divisor of  $p$  and  $a$  is 1, so  $\gcd(p, a) = 1$ .
17. Suppose  $q$  is a natural number greater than 1 with the property that  $q$  divides  $a$  or  $q$  divides  $b$  whenever  $q$  divides  $ab$ . Assume  $q$  is composite. Then  $q$  has a divisor  $m$  that is not 1 and not  $q$ . That is,  $q = mn$  for some integer  $n$ , where  $1 < n < q$ . Then  $q$  divides  $mn$ , so by the given property,  $q$  divides  $m$  or  $q$  divides  $n$ . But  $m$  and  $n$  are less than  $q$ , so this is impossible. Therefore  $q$  is prime.
18. Suppose  $a$  and  $b$  are relatively prime nonzero integers and  $c$  is an integer. Then  $\gcd(a, b) = 1$ , so 1 is a linear combination of  $a$  and  $b$ . That is,  $1 = as + bt$  for some integers  $s$  and  $t$ . Then  $acs + bct = c$ . Thus  $x = cs$ ,  $y = ct$  is an integer solution for the equation  $ax + by = c$ .

19. Let  $a$  and  $b$  be nonzero integers and  $d = \gcd(a, b)$ . Let  $m = b/d$  and  $n = a/d$ . Suppose  $x = s$  and  $y = t$  is a solution for  $ax + by = c$ . Then  $as + bt = c$ . Now let  $k$  be an integer. Then  $a(s + km) + b(t - kn) = as + bt + akm - bkn = c + ak(b/d) - bk(a/d) = c$ . Therefore, for every integer  $k$ ,  $x = s + km$  and  $y = t - kn$  is a solution for  $ax + by = c$ .
20. (a)  $\text{lcm}(6, 14) = 42$ . (b)  $\text{lcm}(10, 35) = 70$ .  
 (c)  $\text{lcm}(21, 39) = 273$ . (d)  $\text{lcm}(12, 48) = 48$ .
21. (a) Part 1. Suppose  $a$  divides  $b$ . Then  $b$  is a common multiple of  $a$  and  $b$ , so condition (i) is satisfied. Now suppose  $n$  is a common multiple. Then  $b$  divides  $n$ , so  $n \leq b$ . Therefore condition (ii) is satisfied, so  $\text{lcm}(a, b) = b$ .  
 Part 2. Suppose  $m = b$ . Then  $b(= m)$  is a multiple of both  $a$  and  $b$ , so  $a$  divides  $b$ .
- (b) Part 1. Suppose  $m = b$ . Then  $b$  is a common multiple of  $a$  and  $b$ , so  $a$  divides  $b$ .  
 Part 2. Suppose  $a$  divides  $b$ . Then  $b$  is a common multiple of  $a$  and  $b$ ; and  $b$  is the smallest multiple of  $b$ , so  $\text{LCM}(a, b) = b$ .
- (c) Suppose  $d = 1$ . Since  $m$  is a multiple of  $b$ ,  $m = bk$  for some integer  $k$ . By part (b)  $m = bk \leq ab$ , so  $k \leq a$ . Since  $m$  is a multiple of  $a$ ,  $a$  divides  $bk$ . By part (c) of Exercise 14,  $a$  divides  $k$ . Thus  $a \leq k$ . Then  $a = k$ , so  $m = ab$ .
- (d) Suppose  $c$  divides  $a$  and  $b$ . Then, since  $a$  divides  $m$ ,  $c$  divides  $m$  as well, so  $\frac{a}{c}$ ,  $\frac{b}{c}$  and  $\frac{m}{c}$  are all natural numbers. Now, since  $(\frac{a}{c})c$  divides  $(\frac{m}{c})c$  and  $(\frac{b}{c})c$  divides  $(\frac{m}{c})c$ , we have by Exercise 7(m) in Section 1.4 that  $\frac{a}{c}$  divides  $\frac{m}{c}$  and  $\frac{b}{c}$  divides  $\frac{m}{c}$ . That is,  $\frac{m}{c}$  is a common multiple of  $\frac{a}{c}$  and  $\frac{b}{c}$ . To see that  $\frac{m}{c}$  is the least common multiple, let  $n$  be another common multiple of  $\frac{a}{c}$  and  $\frac{b}{c}$ . Then  $nc$  is a common multiple of  $a$  and  $b$ . Thus  $nc$  divides  $m$ . Therefore by Exercise 7(m), we see that  $n$  divides  $\frac{m}{c}$ . Therefore  $\text{lcm}(\frac{a}{c}, \frac{b}{c}) = \frac{m}{c}$ .
- (e) Let  $n$  be a natural number.  $n$  divides both  $an$  and  $bn$ , so by (d),  $\text{lcm}(a, b) = \text{lcm}(\frac{an}{n}, \frac{bn}{n}) = \frac{1}{n} \text{lcm}(an, bn)$ . Thus  $\text{lcm}(an, bn) = n \cdot \text{lcm}(a, b)$ .
- (f) By definition,  $d$  divides  $a$  and  $d$  divides  $b$ , so by part (d)  $\text{lcm}(\frac{a}{d}, \frac{b}{d}) = \frac{1}{d} \text{lcm}(a, b)$ . By Exercise 14(d),  $\gcd(\frac{a}{d}, \frac{b}{d}) = 1$ . By part (c)  $\text{lcm}(\frac{a}{d}, \frac{b}{d}) = \frac{a}{d} \cdot \frac{b}{d}$ . Therefore  $m = d \cdot \text{lcm}(\frac{a}{d}, \frac{b}{d}) = d(\frac{a}{d} \cdot \frac{b}{d}) = \frac{ab}{d}$ , so  $d \cdot m = ab$ .
22. (a) F. The claim is false: 125, 521, 215, and other numbers all work.  
 (b) A.  
 (c) F. You cannot prove a statement with an example. Here the “proof” only examines the case where  $x = \pi$ .  
 (d) A.  
 (e) C. The second to the last sentence should read, “Then  $c$  divides 1.” The correct sentence should be justified by previous exercises.  
 (f) A.